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Néron models of jacobians over base schemes of dimension greater than 1

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Abstract. We investigate to what extent the theory of Néron models of jacobians and of abel–jacobi maps extends to relative curves over base schemes of dimension greater than 1. We give a necessary and sufficient criterion for the existence of a Néron model. We use this to show that, in general, Néron models do not exist even after making a modification or even alteration of the base. On the other hand, we show that Néron models *do* exist outside some codimension-2 locus.

1. Introduction

Let C/S be a proper generically smooth curve over a Dedekind scheme. The jacobian J of the generic fibre of C has a Néron model N/S (see [5, 25]). If a point $\sigma \in C(K)$ is given, we obtain a canonical map $\alpha: C^{\text{sm}} \rightarrow N$ from the smooth locus of C/S to the Néron model; this map is called the *abel–jacobi* map. If moreover C/S is for example semistable, then it is possible to give a very explicit description of the Néron model N and the abel–jacobi map α , using the relative Picard functor of C/S (see for example [5, 8]).

Suppose now that we replace the Dedekind scheme S by an arbitrary regular scheme. The main aim of this paper is to understand to what extent the theory of Néron models and abel–jacobi maps described above carries across into this more general situation, and in what ways the theory must be modified. First we give a definition of the Néron model.

Definition 1.1. Let S be a scheme and $U \subseteq S$ a schematically dense open subscheme. Let A/U be an abelian scheme. A *Néron model* for A over S is a smooth separated algebraic space N/S together with an isomorphism $A \rightarrow N_U = N \times_S U$ satisfying the following universal property: *Let $T \rightarrow S$ be a smooth morphism of algebraic spaces and $f: T_U \rightarrow N_U$ any U -morphism. Then there exists a (unique) S -morphism $F: T \rightarrow N$ such that $F|_{T_U} = f$.*

Our non-existence results still hold, with almost the same proofs, if the test objects $T \rightarrow S$ are assumed to be smooth morphisms of schemes rather than of algebraic spaces.

Perhaps a more conventional (though less general) definition would be to assume S to be integral and to have A an abelian scheme over the generic point of S . However, such A would

extend to an abelian scheme over some non-empty open $U \subseteq S$, and an abelian scheme over a regular integral base is automatically the Néron model of its generic fibre. As such, the two definitions are roughly equivalent.

Note that if a Néron model exists, then it is unique up to unique isomorphism, and has a canonical structure as a group algebraic space. What we here call a ‘Néron model’ is more commonly called a ‘Néron *lft* model’ (cf. [5]), where *lft* stands for ‘locally of finite type’, to emphasise that quasi-compactness of N/S is not assumed in the definition.

Let C/S be a semistable curve which is smooth over a dense open subscheme $U \subseteq S$. Write J for the jacobian of the smooth proper curve C_U/U . If C/S is pointed, a Néron model of the jacobian admits an abel–jacobi map from the smooth locus of C/S (by the universal property). Our first result is essentially negative in nature. We say that such a curve C/S is *aligned* if it satisfies a certain rather strong condition described in terms of the dual graphs of fibres and the local structure of the singularities (Definition 2.11).

Theorem 1.2 (Theorem 5.17, Lemma 6.2). *Suppose S is regular.*

- (i) *If the jacobian J/U admits a Néron model over S , then C/S is aligned.*
- (ii) *If C is regular and C/S is aligned, then the jacobian J/U admits a Néron model over S .*

One consequence of the above theorem is that if C/S is not aligned then no proper regular semistable model of C_U over S is aligned. If C/S is split semistable and smooth over the complement of a strict normal crossings divisor in S , then we can determine whether or not a Néron model exists without needing to construct a regular model; see Section 3. We have the following corollary of the above theorem.

Corollary 1.3 (Corollary 6.4). *Let S be an excellent scheme, regular in codimension 1. Let $U \subseteq S$ be a dense open regular subscheme, and let C/S be a semistable curve which is smooth over U . Then there exists an open subscheme $U \subseteq V \subseteq S$ such that the complement of V in S has codimension at least 2 in S and such that the jacobian of C_U/U admits a Néron model over S whose fibrewise-identity-component is an S -scheme.*

It is easy to construct examples (cf. Section 2.3) of non-aligned semistable curves; intuitively, ‘most’ semistable curves over base schemes of dimension greater than 1 are non-aligned, and so do not admit Néron models (one sees easily from the definition that a semistable curve over a Dedekind scheme must be aligned). However, perhaps inspired by [6], one might ask whether C/S admits a Néron model after blowing up S , or making some alteration of S . This turns out again to have a negative answer:

Theorem 1.4 (Proposition 2.22). *Let S be a locally noetherian normal scheme, C/S semistable over S and smooth over a dense open $U \subseteq S$, and $f: S' \rightarrow S$ a proper surjective morphism. Then C/S is aligned if and only if f^*C/S' is aligned.*

Combining with Theorem 1.2 we see that a non-aligned semistable jacobian does not admit a Néron model even after blowing up or altering the base scheme.

For many applications, rather than needing the full strength of the Néron mapping property, one only needs to extend (a multiple of) a single section of the jacobian J to some semi-abelian scheme over S (maybe after an alteration of S). However, it turns out that this does not make the problem easier; in general extending a section in this way is as hard as constructing a Néron model. We illustrate this in Section 7 with an example.

1.1. Idea of the proof: Néron models via the relative Picard functor. Our existence and non-existence results for the Néron model proceed via an auxiliary object, the ‘total-degree-zero relative Picard functor’ $\mathrm{Pic}_{C/S}^{[0]}$ (see Section 5.7), a group-algebraic-space. The base change of $\mathrm{Pic}_{C/S}^{[0]}$ to the locus $U \subseteq S$ where C is smooth coincides with the jacobian of C_U/U . It is also rather easy to show that $\mathrm{Pic}_{C/S}^{[0]}$ satisfies the ‘existence’ part of the Néron mapping property whenever C is regular. However, if C/S has non-irreducible fibres, then $\mathrm{Pic}_{C/S}^{[0]}$ is in general highly non-separated, and for this reason it fails to satisfy the ‘uniqueness’ part of the Néron mapping property. As such, we wish to construct some kind of ‘separated quotient’ of $\mathrm{Pic}_{C/S}^{[0]}$, in the hope that this will satisfy the whole Néron mapping property.

Now the failure of separatedness in a group-space is measured by the failure of the unit section to be a closed immersion; as such, a natural way to construct a ‘separated quotient’ of $\mathrm{Pic}_{C/S}^{[0]}$ is to quotient by the closure of the unit section. Quotients by *flat* subgroup-spaces always exist in the category of algebraic spaces, but the problem is that the closure of the unit section in $\mathrm{Pic}_{C/S}^{[0]}$ (which we shall refer to as \bar{e}) is not in general flat over S , and so this quotient does not exist.¹⁾ By a slightly more delicate argument, one can even show that the existence of a Néron model is *equivalent* to the flatness of \bar{e} . The technical heart of this paper is the proof that the flatness of \bar{e} over S is equivalent to the alignment of C/S (our combinatorial condition introduced in Section 2); this is carried out in Sections 4 and 5.

1.2. Applications. Here we briefly mention four applications of the theory developed in this paper.

Height jumping. In [17], Richard Hain defined an invariant, the height jump, associated to a morphism of variations of mixed Hodge structures, and conjectured that in certain cases this invariant should be non-negative. In [4] we use the theory of Néron models and labelled graphs developed in this paper, along with a study of resistances in electrical networks, to prove some cases of this conjecture. We use the same techniques to extend the definition of the height jump to an algebraic setting. For 1-dimensional families of abelian varieties we show also in [4] that the algebraic height jump is actually *bounded*, and we use this to give a new proof of a theorem of Silverman and Tate [30, 32].

Bounds on orders of torsion points. In [19, 20] we connect the existence of Néron models for families of jacobians to the question of bounding the orders of rational torsion points on abelian varieties, extending the work on algebraic height jumping in [4].

¹⁾ One might be tempted to apply the work [29] of Raynaud and Gruson to flatten \bar{e} by blowing up S , but note that $\mathrm{Pic}_{C/S}^{[0]}$ is not quasi-compact over S , and so their results do not apply; indeed, one consequence of our results is that, if C/S is not aligned, then \bar{e} does not become flat after *any* modification of S .

The case of universal curves. The universal stable pointed curve over the Deligne–Mumford–Knudsen compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli stack of pointed curves is never aligned unless $g = 0$ or $n = g = 1$. As such, a Néron model of the universal jacobian does not in general exist. In [18], we construct a morphism $\beta: \widetilde{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$ over which a Néron model of the universal jacobian does exist, and such that β is universal among morphisms with this property.

Connection to logarithmic geometry. In [3], Bellardini adapts the definition of alignment to the setting of logarithmic geometry, and shows that a family of curves is aligned if and only if it is log cohomologically flat and the logarithmic Picard functor is separated. In particular, this gives a way to verify in examples that a family of curves is log cohomologically flat.

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2. Definition and basic properties of aligned curves

Definition 2.1. A (finite) graph is a triple (V, E, ends) where V and E are finite sets (vertices and edges), and $\text{ends}: E \rightarrow (V \times V)/S_2$ is a function, which we think of as assigning to each edge the unordered pair of its endpoints. A *loop* is an edge whose endpoints are the same. A *circuit* is a path of positive length which starts and ends at the same vertex, and which does not repeat any edges or any other vertices.

Let L be a monoid or set, and Γ a graph. An *edge-labelling* of Γ by L is a function ℓ from the set of edges of Γ to L . We call the pair (Γ, ℓ) a *graph with edges labelled by L* .

We say a graph is *2-vertex-connected* if it is connected and remains connected after deleting any one vertex. When we delete a vertex, we also delete all edges to it. The empty graph is not connected.

Definition 2.2. Let L be a monoid, and (H, ℓ) a 2-vertex-connected graph labelled by L . We say (H, ℓ) is *aligned* if for all pairs of edges e, e' there exist positive integers n, n' such that

$$\ell(e)^n = \ell(e')^{n'}.$$

In other words, non-trivial relations should hold between the labels of edges in H .

Let L be a monoid, and (G, ℓ) a graph labelled by L . We say G is *aligned* if every 2-vertex-connected subgraph²⁾ of G is aligned (equivalently, if every circuit in G is aligned).

2.1. Local structure of semistable curves.

Definition 2.3. Let k be a separably closed field. A *curve over k* is a proper morphism $\pi: C \rightarrow \operatorname{Spec} k$ such that every irreducible component of C/k is of dimension 1. The curve C/k is called *semistable* if it is connected and, for every point $c \in C$, either $C \rightarrow k$ is smooth at c , or c has completed local ring isomorphic to $k[[x, y]]/(xy)$ (i.e. only ordinary double point singularities).

Let S be a scheme. A (semistable) *curve over S* is a proper flat finitely presented morphism $C \rightarrow S$, all of whose fibres over points with values in separably closed fields are (semi-stable) curves.

Remark 2.4. (i) In the classical definition of a semistable curve, one works with fibres over algebraically closed fields, not separably closed fields. This makes our definition appear more restrictive, but it is in fact equivalent to the classical definition, see [23, Proposition 10.3.7].

(ii) Let $f: C \rightarrow S$ be a semistable curve. Since f is flat and has reduced geometric fibres, it is cohomologically flat in dimension 0 (reduce to the noetherian case, then apply [11, Proposition 7.8.6]).

Proposition 2.5. Let S be a locally noetherian scheme, C/S a semistable curve, s a geometric point of S , and c a non-smooth geometric point of C lying over s . We have:

- (i) There exists an element α in the maximal ideal of the completed étale local ring $\widehat{\mathcal{O}_{S,s}^{\text{et}}}$ and an isomorphism of complete local rings

$$\frac{\widehat{\mathcal{O}_{S,s}^{\text{et}}}[[x, y]]}{(xy - \alpha)} \rightarrow \widehat{\mathcal{O}_{C,c}^{\text{et}}}.$$

- (ii) The element α is not in general unique, but the ideal $\alpha \widehat{\mathcal{O}_{S,s}^{\text{et}}} \triangleleft \widehat{\mathcal{O}_{S,s}^{\text{et}}}$ is unique. Moreover, this ideal comes via base change from a unique principal ideal of $\mathcal{O}_{S,s}^{\text{et}}$, which we call the singular ideal of c .

- (iii) Suppose moreover that C is smooth over a schematically dense open $U \hookrightarrow S$. Let $\alpha \in \mathcal{O}_{S,s}^{\text{et}}$ be a generator of the singular ideal. Then there is an isomorphism

$$\frac{\widehat{\mathcal{O}_{S,s}^{\text{et}}}[[x, y]]}{(xy - \alpha)} \rightarrow \widehat{\mathcal{O}_{C,c}^{\text{et}}}.$$

Proof. Write $(A, \mathfrak{m}_A) = \mathcal{O}_{S,s}^{\text{et}}$ and $(B, \mathfrak{m}_B) = \mathcal{O}_{C,c}^{\text{et}}$, and $\rho: A \rightarrow B$ for the canonical map.

- (i) From the definition of a semistable curve, we know that there is an isomorphism

$$\frac{(\hat{A}/\mathfrak{m}_{\hat{A}}\hat{A})[[x, y]]}{(xy)} \rightarrow \hat{B}/\mathfrak{m}_{\hat{A}}\hat{B}.$$

²⁾ That is, a subset $E' \subseteq E$ and $V' \subseteq V$ such that $\text{ends}(e)$ lands in V' for all $e \in E'$.

Let $u, v \in \hat{B}$ be any lifts of x and y , respectively. We see that $uv \in \mathfrak{m}_{\hat{A}}\hat{B}$, and that

$$\mathfrak{m}_{\hat{B}} = u\hat{B} + v\hat{B} + \mathfrak{m}_{\hat{A}}\hat{B}.$$

We now apply [23, Lemma 10.3.20] to find an element $\alpha \in \mathfrak{m}_{\hat{A}}$ and an isomorphism

$$\frac{\hat{A}[[x, y]]}{(xy - \alpha)} \rightarrow \hat{B}.$$

(ii) The uniqueness part is immediate from Lemma 2.8. For the ‘moreover’ part, define

$$F := \text{Fit}_1(\Omega_{B/A}^1/B).$$

The morphism $A \rightarrow B/F$ is unramified and B/F is local, so by [31, Tag 04GL] we find that $B/F = A/I$ for some ideal $I \triangleleft A$, which will be the singular ideal. Write $\alpha \in \hat{A}$ for an element with

$$\frac{\widehat{\mathcal{O}_{S,s}^{\text{et}}}[[x, y]]}{(xy - \alpha)} \xrightarrow{\sim} \widehat{\mathcal{O}_{C,c}^{\text{et}}}.$$

We claim now that $I\hat{A} = \alpha\hat{A}$ as \hat{A} -modules. We establish the claim in two steps:

Step 1: We have

$$\frac{\hat{B}}{F\hat{B}} = \frac{B}{F} \otimes_B \hat{B}$$

which is the completion of the local ring $\frac{B}{F}$ with respect to its maximal ideal. Similarly

$$\frac{\hat{A}}{I\hat{A}} = \frac{A}{I} \otimes_A \hat{A}$$

is the completion of the local ring $\frac{A}{I}$ with respect to its maximal ideal. Hence, as \hat{A} -modules,

$$\frac{\hat{B}}{F\hat{B}} = \frac{\hat{A}}{I\hat{A}}.$$

Step 2: Formation of the relative differentials and of the Fitting ideal commutes with completions, so an easy calculation gives that $F \otimes_B \hat{B} = (x, y)\hat{B}$, and hence

$$\frac{\hat{B}}{F\hat{B}} = \frac{\hat{B}}{(x, y)\hat{B}} = \frac{\hat{A}}{\alpha\hat{A}}$$

as \hat{A} -modules. Combining with Step 1, we find that

$$\frac{\hat{A}}{I\hat{A}} = \frac{\hat{A}}{\alpha\hat{A}},$$

and hence $I\hat{A} = \alpha\hat{A}$ as required.

The two steps show that $I\hat{A} = \alpha\hat{A}$ is a principal ideal of \hat{A} . Since \hat{A} is faithfully flat over A , we deduce by Proposition 2.9 that I is a principal ideal of A . The uniqueness comes again from faithful flatness.

(iii) We let α be as in (i) and choose α' to be a generator of the singular ideal I defined in the proof of (ii). Then α and α' generate the same principal ideal of \hat{A} . By Lemma 2.10 we have that α is not a zerodivisor, and so we deduce that α and α' differ by multiplication by a unit in \hat{A} . We can then easily write down an isomorphism

$$\frac{\hat{A}[[x, y]]}{(xy - \alpha)} \xrightarrow{\sim} \frac{\hat{A}[[x, y]]}{(xy - \alpha')},$$

so we get an isomorphism

$$\frac{\hat{A}[[x, y]]}{(xy - \alpha')} \xrightarrow{\sim} \hat{B}. \quad \square$$

Remark 2.6. A similar argument shows that if C/S is smooth over a schematically dense open, then given a point $s \in S$, a finite separable extension k of the residue field of s , and c a k -valued point of $\text{Sing}(C/S)$ lying over s , we may take the singular ideal of c to be generated by an element in the unique finite étale extension of the henselisation of $\mathcal{O}_{S,s}$ whose residue field is k .

Remark 2.7. If S is not assumed to be locally noetherian, one can define the singular ideal as the pullback of the fitting ideal of the sheaf of relative differentials (the ideal F from the proof of Proposition 2.5). One can then show that this ideal is principal by reducing to the noetherian case. However, most of our results will require that S is even regular, so there seems little to be gained from such additional generality at this stage.

Lemma 2.8. *Let (A, \mathfrak{m}_A) be a local ring, $\alpha, \beta \in \mathfrak{m}_A$ elements, and*

$$\varphi: \frac{A[[x, y]]}{(xy - \alpha)} \rightarrow \frac{A[[s, t]]}{(st - \beta)}$$

an isomorphism of A -algebras. Then $\alpha A = \beta A$.

Proof. Write

$$R := \frac{A[[x, y]]}{(xy - \alpha)} \quad \text{and} \quad R' := \frac{A[[s, t]]}{(st - \beta)}.$$

The first fitting ideal of the module of continuous relative differentials $\Omega_{R/A}$ is the ideal $(x, y) \triangleleft R$, and similarly

$$\text{Fit}_1(\Omega_{R'/A}^1/R') = (s, t) \triangleleft R'.$$

Since R and R' are by assumption isomorphic as A modules, the same is true of the quotients

$$R/\text{Fit}_1(\Omega_{R/A}^1/R) \quad \text{and} \quad R'/\text{Fit}_1(\Omega_{R'/A}^1/R'),$$

so we see that

$$A/\alpha = R/(x, y) \quad \text{and} \quad A/\beta = R'/(s, t)$$

are isomorphic as A -modules, hence $\alpha A = \beta A$. \square

Lemma 2.9. *Let $\varphi: S' \rightarrow S$ be a flat morphism of schemes, and $s' \in S'$ be a point lying over some $s \in S$. Let $D \hookrightarrow S$ be a closed subscheme. Then the ideal sheaf of D can be generated near s by a single (regular) element if and only if the ideal sheaf of the pullback $D \times_S S' \hookrightarrow S'$ can be generated near s' by a single (regular) element.*

Proof. This is well known and is omitted. \square

Lemma 2.10. *Let $\pi: C \rightarrow S$ be semistable, and assume that $C \rightarrow S$ is smooth over some schematically dense open subscheme $U \subseteq S$. Let k be a separably closed field, and $c \in C(k)$ be a point lying over a point $s \in S(k)$ such that c is not in the smooth locus of $C \rightarrow S$. Suppose we have an element $\alpha \in \widehat{\mathcal{O}_{S,s}^{\text{et}}}$ and an isomorphism*

$$\frac{\widehat{\mathcal{O}_{S,s}^{\text{et}}}[[x, y]]}{(xy - \alpha)} \rightarrow \widehat{\mathcal{O}_{C,c}^{\text{et}}}.$$

Then α is a non-zerodivisor in $\widehat{\mathcal{O}_{S,s}^{\text{et}}}$.

Proof. We may assume that $S = \text{Spec } R$ with R complete and strictly local; U remains schematically dense by [13, Théorème 11.10.5 (ii)]. Let Z be the closed subscheme of S cut out by α . From our smoothness assumption, it follows that $Z \times_S U$ is empty.

Now we will assume that α is a zerodivisor in R , and derive a contradiction by showing that $Z \times_S U$ is *non* empty. First, note that Z contains an associated prime of R ; indeed, the union of the associated primes of R is exactly equal to the set of zerodivisors in R . However, we claim that U must contain every associated prime of R . To see this, note that by [31, Tag 083P] U is dense in S and contains every embedded prime, hence it contains every associated prime (since every associated prime is either embedded, or is the generic point of an irreducible component). \square

2.2. The definition of an aligned curve. The most important definition in this paper is the following:

Definition 2.11. Let S be a locally noetherian scheme and C/S a semistable curve. Let $s \in S$ be a geometric point, and write $\mathcal{O}_{S,s}^{\text{et}}$ for the étale local ring of S at s . Write Γ for the dual graph (defined as in [23, Definition 10.3.17]) of the fibre C_s . Let L_s be the monoid of principal ideals of $\mathcal{O}_{S,s}^{\text{et}}$. We label Γ by elements of L_s by assigning to an edge $e \in \Gamma$ the singular ideal $l \in L_s$ of the singular point of C_s associated to e (cf. Proposition 2.5).

We say C/S is *aligned at s* if and only if this labelled graph is aligned. We say C/S is *aligned* if it is aligned at s for every geometric point s of S .

Remark 2.12. (i) Let $f: T \rightarrow S$ be any morphism. Let t be a geometric point of T , lying over a geometric point s of S . Then the labelled graph Γ_t of C_T over t has the same underlying graph as that of C over s , and the labels on Γ_t are obtained by pulling back those on Γ_s along f . One can see this for example by using the construction of the singular ideal in terms of the fitting ideal of the sheaf of relative differentials, whose formation is well-behaved under base change.

(ii) The property of ‘being aligned’ is fppf-local on the target, i.e., it is preserved under flat base change and satisfies fppf descent.

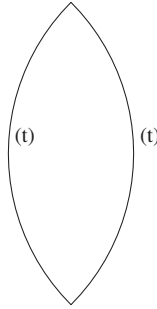
(iii) Given S excellent, integral and regular in codimension 1, and C/S semistable and generically smooth, one sees easily that there exists an open subscheme $V \subseteq S$ such that C_V/V is aligned and such that the closed subscheme $S - V$ has codimension at least 2 in S .

2.3. Examples of aligned and non-aligned curves. Let $S = \operatorname{Spec} \mathbb{C}[[s, t]]$. We give two semistable curves C_1, C_2 over S , with the same closed fibre, and with C_1 aligned and C_2 non-aligned.

Example 2.13 (The aligned curve C_1). We define C_1 to be the S -scheme cut out in weighted projective space $\mathbb{P}_S(1, 1, 2)$ (with affine coordinates x, y) by the equation

$$y^2 = ((x - 1)^2 - t)((x + 1)^2 + t).$$

This is naturally the pullback of the curve over $\operatorname{Spec} \mathbb{C}[[t]]$ defined by the same equation inside $\mathbb{P}_{\mathbb{C}[[t]]}(1, 1, 2)$ along the natural map $S \rightarrow \operatorname{Spec} \mathbb{C}[[t]]$. The closed fibre is a 2-gon, and the labelled dual graph is

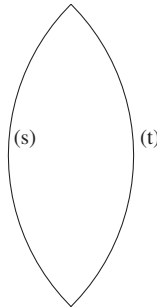


The relation $(t) = (t)$ holds, implying that C_1/S is aligned (there is only one circuit in the dual graph, with only two edges, so there is only one condition to check).

Example 2.14 (The non-aligned curve C_2). We define C_2 to be the S -scheme cut out in weighted projective space $\mathbb{P}_S(1, 1, 2)$ (with affine coordinates x, y) by the equation

$$y^2 = ((x - 1)^2 - s)((x + 1)^2 + t).$$

This does not arise as the pullback of a curve over any trait. The closed fibre is a 2-gon, and the labelled dual graph is



There do not exist positive integers m and n such that the relation $(s)^m = (t)^n$ holds. As (s) and (t) appear on the same circuit, this implies that the curve is not aligned.

Remark 2.15. Let C/S be a semistable curve, and let s be a geometric point of S . Suppose firstly that the fibre over s is 2-vertex-connected (for example an n -gon for some n). Then C/S being aligned at s means roughly that C/S looks (locally at s) as if it is the pullback of a semistable curve over some trait (this is only a heuristic, and cannot be used as a substitute definition). Suppose on the other hand that C/S is of compact type (i.e. the dual graphs of

all geometric fibres are trees, or equivalently the jacobian $\text{Pic}_{C/S}^0$ is an abelian scheme); then C/S is automatically aligned. In both of these situations, it seems perhaps plausible that a Néron model can exist. In general, being aligned can be thought of as some kind of common generalisation of these two situations.

2.4. Local-global for relations between divisors. This section contains various lemmas we will need for proving that both being aligned and being non-aligned are preserved under alterations (Proposition 2.22).

Lemma 2.16. *Let X be a connected noetherian scheme, and $p, q \in X$. Then there exist points $x_1, \dots, x_n \in X$ with $x_1 = p$, $x_n = q$, and for all i , either $x_i \in \overline{\{x_{i+1}\}}$ or $x_{i+1} \in \overline{\{x_i\}}$.*

Proof. Since X is noetherian, it has finitely many irreducible components, so we are done by [31, Tag 0904]. \square

We say a monoid M is *cyclic* if it can be generated by a single element.

Definition 2.17. Let \mathfrak{S} denote the set of submonoids of the monoid $\mathbb{N}_0 \times \mathbb{N}_0$, where $\mathbb{N}_0 := \{n \in \mathbb{Z} \mid n \geq 0\}$. Given an element $M \in \mathfrak{S}$, we define the *saturation* of M to be the set of all $m \in \mathbb{N}_0 \times \mathbb{N}_0$ such that $am \in M$ for some $a \in \mathbb{N}_{>0}$. We remark that if M is cyclic then so is its saturation.

Given a scheme X and two effective Cartier divisors D, E on X , we define a function

$$\zeta = \zeta_{D,E}: X \rightarrow \mathfrak{S}, \quad x \mapsto \{(m, n) \mid mD = nE \text{ locally at } x\}.$$

Note that a relation holds Zariski-locally if and only if it holds fppf-locally.

Lemma 2.18. *If $x \in \overline{\{y\}}$ then $\zeta(x) \subseteq \zeta(y)$.*

Proof. Say $mD = nE$ holds locally at x . Then the same relation holds on some Zariski neighbourhood U of x , and U is also a Zariski neighbourhood of y . \square

Lemma 2.19. *Let $x \in X$ lie in the union of the supports of D and E . Then the saturation of the monoid $\zeta(x)$ is cyclic.*

Proof. Suppose not, and let (m_1, n_1) and (m_2, n_2) be elements of $\zeta(x)$ which are not contained in the same cyclic submonoid. Then (working locally at x) we have that $m_1 D = n_1 E$ and $m_2 D = n_2 E$, hence $n_1 m_2 E = m_1 n_2 E$.

Now let $f \in \mathcal{O}_{X,x}$ be a defining equation for E . Because E is Cartier, f cannot be a zerodivisor. Because $n_1 m_2 E = m_1 n_2 E$, we have that

$$(f^{n_1 m_2}) = (f^{m_1 n_2})$$

as ideals of $\mathcal{O}_{X,x}$. Because f is not a zerodivisor, we deduce that there exists $u \in \mathcal{O}_{X,x}^\times$ such that $f^{n_1 m_2} = u f^{m_1 n_2}$. Therefore $f^{|n_1 m_2 - m_1 n_2|} \in \mathcal{O}_{X,x}^\times$, so either $n_1 m_2 = m_1 n_2$ or E is 0 at x .

Applying the same argument to D , we find that either $n_1 m_2 = m_1 n_2$ or D is 0 at x . Since x is in the union of the supports of D and E , it must be that $n_1 m_2 = m_1 n_2$. It follows that the saturation of $\zeta(x)$ is cyclic. \square

Lemma 2.20. *Let $x, y \in X$ and assume that*

- (i) $x \in \overline{\{y\}}$;
- (ii) x and y both lie in the union of the supports of D and E ;
- (iii) $\zeta(x)$ and $\zeta(y)$ are both unequal to $\{(0, 0)\}$.

Then $\zeta(x)$ and $\zeta(y)$ have the same saturations.

Proof. We have $\zeta(x) \subseteq \zeta(y)$ (Proposition 2.18), and so the same holds for the saturations. By Proposition 2.19 both the saturations are cyclic and hence are equal. \square

Combining the above results, we obtain the following lemma.

Lemma 2.21. *Let X be a noetherian scheme, and let D, E be as above and such that $\text{Supp } D \cup \text{Supp } E$ is connected. Suppose that for all $x \in X$ we have that $\zeta(x) \neq \{(0, 0)\}$ (i.e., non-trivial relations hold between D and E everywhere locally). Then there exist integers m, n with $(m, n) \neq (0, 0)$ and $mD = nE$ (globally on X).*

Proof. Let $S := \text{Supp } D \cup \text{Supp } E$. By quasi-compactness, we find

- (i) a finite open cover U_i ($i = 1, \dots, n$) of X with $U_i \cap S$ non-empty for all i ;
- (ii) for each i , a point $p_i \in U_i \cap S$;
- (iii) for each i , integers m_i, n_i with $(m_i, n_i) \neq (0, 0)$ such that for all i , $n_i D = m_i E$ holds on U_i .

We want to show that some non-trivial relation holds between D and E on the whole of X . For any i and j , we find a chain of specialisations joining p_i and p_j (by Lemma 2.16), and so the saturations of $\zeta(p_i)$ and $\zeta(p_j)$ coincide by Proposition 2.20.

Now if finitely many cyclic monoids in $\mathbb{N}_0 \times \mathbb{N}_0$ all have the same saturation and are not equal to $\{(0, 0)\}$, then the monoids must contain a common non-zero element. This element is the relation we wanted. \square

2.5. Descent of alignment along proper surjective maps. The next theorem is the goal of this section; it shows that being aligned is preserved under and descends along proper surjective base change (e.g. under alterations).

Theorem 2.22. *Let S be locally noetherian and normal, and let $C \rightarrow S$ be semistable and smooth over some schematically dense open $U \subseteq S$. Let $f: S' \rightarrow S$ be a proper surjective map. Write C'/S' for the pullback of C . Then C/S is aligned if and only if C'/S' is aligned.*

Proof. Suppose C/S is aligned, and let s' and s be geometric points of S' and S , respectively, with s' lying over s . There is a natural isomorphism of graphs $\Gamma_s \rightarrow \Gamma_{s'}$, and the labels on $\Gamma_{s'}$ are given by pulling back those on Γ_s . It is then easy to check that alignment of C/S implies alignment of C'/S' .

Conversely, suppose C'/S' is aligned. We want to show that the same holds for C/S . We may assume that S is strictly henselian local, with geometric closed point s . Shrinking S' ,

we may assume that $f^{-1}U$ is dense in S' . Fix a circuit γ in the dual graph of the fibre over s , and two labelling divisors D and E that appear in that circuit – these are divisors since the labels are non-zero-divisors by Lemma 2.10. We will show that some non-trivial relation holds between D and E .

Considering the Stein factorisation of $S' \rightarrow S$, and the fact that S is strictly henselian, we may assume that the fibre S'_s is connected [31, Tag 03QH]. Note also that every irreducible component of each of f^*D , f^*E meets the closed fibre; indeed, let Z be such an irreducible component, then $f(Z)$ is closed (since f is proper) and so contains the closed point s of S , hence Z meets the closed fibre $f^{-1}(s)$. These facts together imply that the union $\text{Supp } f^*D \cup \text{Supp } f^*E$ of the supports of the pullbacks f^*D , f^*E is also connected.

Now by the assumption that C'/S' is aligned we find that, locally at every point in S' , a non-trivial relation holds between f^*D and f^*E . Indeed, since S is local, it is enough to check this at points of S' lying over $s \in S$. Let s' be such a point, with \bar{s}' an associated geometric point. Then the (unlabelled) graph of C'/S' at \bar{s}' is naturally identified with the (unlabelled) graph of C over s , and the labels on the graph over \bar{s}' are just given by pulling back. In particular, the restrictions of f^*D and f^*E to the étale local ring of S' at \bar{s}' correspond to two edges on the same circuit, and so a non-trivial multiplicative relation holds between them by the assumption that C'/S' is aligned. This descends to a multiplicative relation on some étale neighbourhood of \bar{s}' by a finite presentation argument.

A-priori this does not imply that a non-trivial relation holds globally on S' , but using connectedness, we can apply Lemma 2.21 to deduce that, for some $(m, n) \neq (0, 0)$ we have $mf^*D = nf^*E$ globally on S' . By Lemma 2.23 this implies that the same non-trivial relation holds on S : we have $mD = nE$. \square

We show that the pullback map on Cartier divisors (*not* up to linear equivalence) is injective:

Lemma 2.23. *Let S be a locally noetherian normal scheme, and $f: S' \rightarrow S$ a proper morphism whose scheme theoretic image is S (i.e., f does not factor via a non-trivial closed immersion). Let $U \subseteq S$ be a schematically dense open subscheme such that $f^{-1}U$ is schematically dense in S' . Let D and E be effective Cartier divisors on S , supported on $S \setminus U$ (so f^*D and f^*E are again Cartier divisors) and such that $f^*D = f^*E$. Then $D = E$.*

Proof. Because S is normal, the effective Cartier divisors D and E are uniquely determined by the generic points (together with multiplicities) of their associated closed subschemes [14, Proposition 21.7.2]. We may thus reduce to the case where S is the spectrum of a discrete valuation ring R with uniformiser r , and D and E are given by the vanishing of some powers of r , say $D = (r^n)$ and $E = (r^m)$.

By the condition that $f^{-1}U$ is schematically dense in S' , we see that f^*r is regular, and hence that f^*r is regular in $A := H^0(S', \mathcal{O}_{S'})$. The condition that $f^*D = f^*E$ is then equivalent to the condition that f^*r^n and f^*r^m differ by multiplication by a unit in A . Let $a \in A$ be such a unit, so $a \cdot f^*r^m = f^*r^n$. Using that f^*r is regular, we deduce that $a = f^*r^{n-m}$, and hence that either $m = n$ (in which case $D = E$ and we are done) or that f^*r is a unit in A . However, f^*r cannot be a unit since f is surjective. \square

Example 2.24. Here we give an example with the normalisation of a nodal cubic to show that the assumption that S be normal is necessary. Let k be a field, let $A_0 = k[t]$, and let $R_0 = k[t^2, t^3 - t] \subseteq A_0$. Let R and A be the rings obtained from R_0 and A_0 by inverting t^2 , and set $S = \operatorname{Spec} R$ and $S' = \operatorname{Spec} A$. Define Cartier divisors D and E on S by $D = (t^3 - t)$ and $E = (t^2 - 1)$. Then D and E are not equal, but their pullbacks to S' are equal since $t(t^2 - 1) = t^3 - t$ and t is a unit in A .

3. Changing the model of C

Definition 3.1. Let S be a scheme, and $U \subseteq S$ a schematically dense open subscheme. Let C/U be a smooth proper curve. A *model* of C over S is a proper flat curve \tilde{C}/S together with a U -isomorphism $(\tilde{C})_U \rightarrow C$.

From Theorem 1.2 we know that

- if C has a semistable non-aligned model over S , then the jacobian $\operatorname{Jac}_{C/U}$ has no Néron model over S ;
- if C has a regular semistable aligned model over S , then the jacobian $\operatorname{Jac}_{C/U}$ does have a Néron model over S .

What can we say about the existence of a Néron model if we are presented with a semistable aligned model of C which is not regular? We are unable to answer this question in complete generality as our method of producing Néron models for the jacobian requires a regular model of C , which is not known to exist. On the other hand, suppose that U is the complement of a divisor with strict normal crossings and that \tilde{C}/S is split semistable in the sense of [21, §2.22].³⁾ Then we will see in this section how to determine from the labelled graphs of \tilde{C}/S whether or not the jacobian admits a Néron model.

Before going to the general theory, let us consider an example. Let $S = \operatorname{Spec} \mathbb{C}[[u, v]]$, and define $\tilde{C} \subseteq \mathbb{P}_S(1, 1, 2)$ by the affine equation

$$y^2 = ((x - 1)^2 - uv)((x + 1)^2 + uv).$$

This is split semistable and smooth over the open subset $U := S \setminus (uv = 0)$. The labelled graph of \tilde{C} at the closed point of S is a 2-gon, and both edges are labelled by (uv) . As such, the graph is aligned, but clearly \tilde{C} is not regular. After some blowups we obtain a new model \tilde{C} of $C := \tilde{C}_U$, with labelled graph over the closed point of S a 4-gon with two labels equal to (u) and two labels equal to (v) . Clearly this is not aligned, and so the jacobian of C does not admit a Néron model over S .

To generalise this calculation, we need some definitions.

Definition 3.2. Let G be a set and $M(G)$ be the free (commutative) monoid on G . Given a graph Γ with an edge-labelling by non-zero elements of $M(G)$, we say Γ is *regular* if every label lies in G . A *refinement* of Γ is a new $M(G)$ -labelled graph Γ' obtained from Γ by replacing an edge with label $\prod_{g \in G} g^{\epsilon_g}$ by a chain of $\sum_{g \in G} \epsilon_g$ edges, with ϵ_g of those edges labelled by g for every g in G .

³⁾ That is, for all field-valued fibres of \tilde{C}/S we have that all the irreducible components are geometrically irreducible and smooth, and that all singular points are rational.

We do not specify the order of the labels on the new edges, so refinements are in general very non-unique. It is clear that every graph Γ with finitely many edges becomes regular after a finite sequence of refinement operations.

Definition 3.3. We say an $M(G)$ -labelled graph Γ' is a *regularisation* of Γ if Γ' is regular and can be obtained from Γ by a finite sequence of refinements.

Definition 3.4. Let S be a regular scheme, and C/S a semistable curve. Let s be a geometric point of S . Let Γ_s be the labelled graph of C over s , so its labels lie in $\widehat{\mathcal{O}_{S,s}^{\text{et}}} / (\widehat{\mathcal{O}_{S,s}^{\text{et}}})^\times$, which is the free abelian monoid on the height 1 prime ideals of $\mathcal{O}_{S,s}^{\text{et}}$ (since S is regular). We say C/S is *strictly aligned at s* if one (equivalently all) regularisations of Γ_s is aligned. We say C/S is *strictly aligned* if it is so at every geometric point of S .

Remark 3.5. (i) If C/S is strictly aligned, then it is aligned, by Theorem 1.2.

(ii) If C is regular, then all the graphs Γ_s are automatically regular, so that C/S is aligned if and only if it is strictly aligned.

Proposition 3.6. *Let S be a regular excellent separated scheme, let $U \subseteq S$ be a dense open subscheme which is the complement of a strict normal crossings divisor. Let C/S be a split semistable curve which is smooth over U . Then the jacobian of C_U/U has a Néron model over S if and only if C/S is strictly aligned.*

Proof. By [21, Proposition 3.6] we know that C/S admits a resolution of singularities; more precisely, there exists a proper S -morphism $\tilde{C} \rightarrow C$ which is an isomorphism over U and such that \tilde{C} is split semistable and regular. Following through the steps in de Jong's proof, we find that for all geometric points s of S , the labelled graph of \tilde{C} over s is a regularisation of the labelled graph of C over s .

Now if C/S were strictly aligned, then by definition these regularisations are aligned, so a Néron model exists by Theorem 1.2. Conversely, suppose that a Néron model exists. Then again by Theorem 1.2 we have that \tilde{C}/S is aligned, so C/S is strictly aligned. \square

4. Classification of vertical Cartier divisors on certain complete local rings

Let (R, \mathfrak{m}_R) be a regular \mathfrak{m}_R -adically complete local ring. Let $r \in R$ be a non-zero element. Let $A = R[[x, y]]/(xy - r)$. Then A is R -flat and is a complete normal local noetherian domain [12, Scholie 7.8.3, p.215]. Our aim is to classify the principal ideals of A which become trivial after base change over R to $K := \text{Frac } R$. More precisely, we will show:

Theorem 4.1. *Let $a \in A$ be an element such that $a \otimes 1$ is a unit in $A \otimes_R K$. Then there exist*

- an element $s \in R$,
- non-negative integers m, n such that $mn = 0$,
- a unit $u \in A^\times$,

such that $a = sx^n y^m u$.

The proof will occupy the remainder of this section. This result is only relevant for showing that the jacobians of non-aligned curves do not admit Néron models; it is not required for the implication ‘aligned \Rightarrow existence of Néron model’.

The author is very grateful to one of the referees for providing a new proof, based on the original one but half as long and far more readable.

We begin with some lemmas to reduce to the case where the element a is a polynomial. Write \tilde{A} for the localisation of $R[x, y]/(xy - r)$ at the maximal ideal $(x, y) + \mathfrak{m}_R$, so that A is the completion of \tilde{A} at its maximal ideal, and write $\varphi: \tilde{A} \rightarrow A$ for the canonical injection. Note that \tilde{A} is again normal, again by [21, §4.21]. Write $\text{Div } A$ for the free group generated by height 1 prime ideals of A (Weil divisors of $\text{Spec } A$), and write $\text{Div}_F A$ for the subgroup generated by height 1 primes \mathfrak{p} such that $\mathfrak{p} \otimes_R K \cong A \otimes_R K$ (‘fibrals’). We define $\text{Div } \tilde{A}$ and $\text{Div}_F \tilde{A}$ analogously.

Lemma 4.2. *The map $\varphi^*: \text{Div}_F A \rightarrow \text{Div}_F \tilde{A}$ is well-defined and an isomorphism.*

Proof. Fibrals on A and on \tilde{A} are exactly those height 1 primes whose preimages in R are also of height 1 (for dimension reasons). From this we deduce easily that the map is well defined. Since \tilde{A} is regular in codimension 1, we deduce by faithful flatness of the completion that φ^* is injective. It remains to show that it is surjective.

For this, it is enough to show that φ^* induces a surjection on the fibrals. Note that if a fibral prime p of A lies over a prime q of R then the same holds for $\varphi^* p$. Now let q be a height 1 prime ideal of R . If $r \notin q$ then qA and $q\tilde{A}$ are fibrals of height 1. If $r \in q$ then there are exactly two (fibrals) height 1 prime ideals of A above q , namely $q + xA$ and $q + yA$, and similarly for \tilde{A} . In either case, restricted to fibrals lying over q , the map φ^* is an injective map of finite sets of the same cardinality, and so is also surjective. \square

Lemma 4.3. *Let $a \in A$ be an element such that $a \otimes 1$ is a unit in $A \otimes_R K$. Then there exists $\tilde{a} \in \tilde{A}$ and a unit $u \in A^\times$ such that $a = u\tilde{a}$.*

Proof. By Lemma 4.2 we find that there is a Weil divisor D on $\text{Spec } \tilde{A}$ such that $\varphi^* D = \text{div}_A a$. Identify D with its associated closed subscheme of S as in [14, §21.7.1]. Then D satisfies the condition (S_1) (see [14, Proposition 21.7.2]), and the same holds for $\varphi^* D$ by [12, Scholie 7.8.3, p. 215]. Moreover, $\varphi^* D$ is equal to the closed subscheme $V(a) \hookrightarrow S$ cut out by a at all points of S of codimension 1 (using that S is normal and thus regular in codimension 1). Moreover, $V(a)$ satisfies (S_1) (again using that S is normal), so applying [14, Proposition 21.7.2] again, we find that $\varphi^* D = V(a)$ as closed subschemes of S . In particular, $\varphi^* D$ is generated by a single element, so by flat descent the same holds for D , say $D = V(\tilde{a})$. Then $a = u\tilde{a}$ for some unit $u \in \tilde{A}$ as required. \square

Proof of Theorem 4.1. The proof is divided into several steps.

Step 1: From Corollary 4.3 we easily see that there exists $\tilde{a} \in R[x, y]$ and a unit $u \in A^\times$ such that $a = u\tilde{a}$. As such, we may and will assume that a is contained in the image of $R[x, y]$ in A .

Step 2: Let W denote the set of series $\sum_{i \in \mathbb{Z}} w_i T^i$ where $w_i \in K$ and T is a formal variable. The set W is naturally an abelian group under addition of series, but does not have a natural ring structure. It is a module over the ring $K[T]$, with action the usual multiplication.

Let

$$F^+ := \left\{ \sum_{i \in \mathbb{Z}} a_i T^i \mid \text{for all } i \geq 0, a_i \in R \text{ and } a_{-i} \in r^i R \right\}.$$

This set F^+ has a natural multiplication since R is complete with respect to its maximal ideal and hence is also r -adically complete. In what follows we will identify A with F^+ via the R -algebra isomorphism

$$A = \frac{R[[x, y]]}{(xy - r)} \rightarrow F^+, \quad x \mapsto T, \quad y \mapsto r/T.$$

If \mathfrak{p} is a height 1 prime ideal of R , we write $v_{\mathfrak{p}}$ for the normalised valuation associated with \mathfrak{p} . If $v_{\mathfrak{p}}(r) > 0$ then we define $A(\mathfrak{p})$, $W(\mathfrak{p})$ and $F^+(\mathfrak{p})$ in the same way as A , W and F^+ but replacing R by the completed localisation $\hat{R}_{\mathfrak{p}}$. We obtain the following commutative diagram of R -algebras:

$$\begin{array}{ccc} A = \frac{R[[x, y]]}{(xy - r)} & \xrightarrow{\sim} & F^+ \\ \downarrow & & \downarrow \\ A(\mathfrak{p}) = \frac{\hat{R}_{\mathfrak{p}}[[x, y]]}{(xy - r)} & \xrightarrow{\sim} & F^+(\mathfrak{p}). \end{array}$$

Let Ω be the set of height 1 prime ideals of R , and $\Omega_r = \{\mathfrak{p} \in \Omega \mid v_{\mathfrak{p}}(r) > 0\}$.

Step 3: By Step 1, it is enough to consider an element $a \in R[T, r/T] \subseteq F^+ = A$ (here we identify F^+ with A as in Step 2) and show it can be written as $a = sT^n u$ with $s \in R$, $n \in \mathbb{Z}$ and $u \in A^\times$. To prove this, we may and do assume that a is of the form

$$a = \sum_{i=0}^m a_i T^i \in R[T] \subseteq F^+, \quad \text{with } a_0 a_m \neq 0 \text{ and } \gcd(a_0, \dots, a_m) = 1.$$

Since R is regular, it is a UFD, and so the gcd makes sense.

For each $\mathfrak{p} \in \Omega_r$, let $\text{NP}(a; \mathfrak{p})$ be the Newton polygon of a with respect to $v_{\mathfrak{p}}$; here a is viewed as an element in $\hat{R}_{\mathfrak{p}}[[T]] \subseteq F^+(\mathfrak{p})$. We claim that for every $\mathfrak{p} \in \Omega_r$, $\text{NP}(a; \mathfrak{p})$ does not contain any side of slope in $(-v_{\mathfrak{p}}(r), 0)$. Otherwise, pick a $\mathfrak{p} \in \Omega_r$ such that the latter assertion does not hold. The theory of Newton polygons (cf. [1], especially the last paragraph of p. 43) tells us that the polynomial $a = a(T)$ has a root $x_0 \in \overline{K}_{\mathfrak{p}}$ with \mathfrak{p} -adic valuation in $(0, v_{\mathfrak{p}}(r))$; here $K_{\mathfrak{p}}$ denotes the completion of K with respect to the valuation $v_{\mathfrak{p}}$, and $\overline{K}_{\mathfrak{p}}$ its algebraic closure. Then $y_0 := r/x_0 \in \overline{K}_{\mathfrak{p}}$ has \mathfrak{p} -adic valuation > 0 , and so the pair (x_0, y_0) defines a $\tilde{K}_{\mathfrak{p}}$ -valued point of the closed subscheme $V(a \otimes 1) \hookrightarrow \text{Spec}(A(\mathfrak{p}) \otimes_{\hat{R}_{\mathfrak{p}}} K_{\mathfrak{p}})$. It follows that the closed subscheme $V(a \otimes 1) \hookrightarrow \text{Spec}(A \otimes_R K)$ is not empty, contradicting our assumption that $a \otimes 1$ is a unit in $A \otimes_R K$.

Step 4: For each $\mathfrak{p} \in \Omega_r$, let $N_{\mathfrak{p}}$ denote the smallest integer in $[0, m]$ such that $v_{\mathfrak{p}}(a_{N_{\mathfrak{p}}}) = 0$. Assume for the moment that integer $N_{\mathfrak{p}}$ does not depend on $\mathfrak{p} \in \Omega_r$ (we shall denote this integer by N), and that $a_N \in R^\times$. Under these extra assumptions, we claim that $a = T^N \cdot (\text{unit})$. Indeed, by Step 3, we have that $v_{\mathfrak{p}}(a_{N-i}) \geq i \cdot v_{\mathfrak{p}}(r)$ for every $\mathfrak{p} \in \Omega$ and $0 \leq i \leq N$. Hence $a_{N-i}/r^i \in R$ for $0 \leq i \leq N$. Write $a_{N-i} = r^i a'_{N-i}$. We find

$$a = \sum_{i=0}^N r^i a'_{N-i} T^i + \sum_{j>N} a_j T^j = x^N \left(a_N + \sum_{i=1}^N a'_{N-i} y^i + \sum_{j>N} a_j x^{j-N} \right).$$

Since $a_N \in R^\times$, we see that

$$u := a_N + \sum_{i=1}^N a'_{N-i} y^i + \sum_{j>N} a_j x^{j-N} \in A^\times$$

as required.

Step 5: It remains to show that, under the assumption that $a \otimes 1 \in (A \otimes_R K)^\times$, the extra assumptions in Step 4 are always satisfied. If not, there are two possibilities:

- There exist prime ideals $\mathfrak{p}, \mathfrak{q} \in \Omega_r$ such that $N_{\mathfrak{p}} \neq N_{\mathfrak{q}}$. Without loss of generality we assume $i_0 := N_{\mathfrak{q}} > N_{\mathfrak{p}} =: N$. In particular, $0 = v_{\mathfrak{q}}(a_{i_0}) < v_{\mathfrak{q}}(a_N)$.
- All the $N_{\mathfrak{p}}$ are equal but $a_N \notin R^\times$. In this case we choose $\mathfrak{p} \in \Omega_r$ and set $N = N_{\mathfrak{p}}$. Let $\mathfrak{q} \in \Omega$ be such that $v_{\mathfrak{q}}(a_N) \neq 0$. As $\gcd(a_0, \dots, a_m) = 1$ there exists $i_0 \neq N$ such that $v_{\mathfrak{q}}(a_{i_0}) = 0$. Up to the reflection substitution $T \mapsto r/T$ we may assume that $i_0 > N$.

In both cases, we get $\mathfrak{p} \in \Omega_r$, $\mathfrak{q} \in \Omega$ and $i_0 > N = N_{\mathfrak{p}}$ such that $v_{\mathfrak{q}}(a_N) > v_{\mathfrak{q}}(a_{i_0}) = 0$. Furthermore, we may assume $v_{\mathfrak{p}}(a_{i_0}) = 0$. Indeed, if $v_{\mathfrak{p}}(a_{i_0}) > 0$, consider the element

$$a(1 + a_N T^{i_0-N}) =: \sum_{j=0}^{m+1} b_j T^j.$$

One checks that for all $1 \leq i \leq N$ we have $v_{\mathfrak{p}}(b_{N-i}) \geq i \cdot v_{\mathfrak{p}}(r)$ and $v_{\mathfrak{p}}(b_N) = 0$. Further, we have $b_{i_0} = a_{i_0} + a_N^2$. Therefore $v_{\mathfrak{p}}(b_{i_0}) = 0$ since a_N is a \mathfrak{p} -unit, and $v_{\mathfrak{q}}(b_{i_0}) = 0$ since $v_{\mathfrak{q}}(a_{i_0}) = 0$ and $v_{\mathfrak{q}}(a_N) > 0$. Finally, we still have $v_{\mathfrak{q}}(b_N) > 0$ since b_N is either a_N (if $i_0 - N > N$) or $a_N(1 + a_{2N-i_0})$ (if $i_0 - N \leq N$). So up to replacing a by $a(1 + a_N T^{i_0-N})$ we may always assume that a_{i_0} is a \mathfrak{p} -unit.

Now we claim that such an element a is not invertible in $A \otimes K$. Otherwise, denote its inverse in $A \otimes K$ by a^{-1} . Define $f \in A(\mathfrak{p})$ by

$$-f = \frac{a_0}{a_N} T^{-N} + \dots + \frac{a_{N-1}}{a_N} T^{-1} + \frac{a_{N+1}}{a_N} T + \dots + \frac{a_m}{a_N} T^{m-N}.$$

One checks easily (cf. Step 4) that the series $\sum_{i \geq 0} f^i$ converges to an element of $A(\mathfrak{p})$, and that in $A(\mathfrak{p}) \otimes K_{\mathfrak{p}}$ we have

$$aa_N^{-1} T^{-N} \sum_{i \geq 0} f^i = 1.$$

Now $A(\mathfrak{p}) \otimes K_{\mathfrak{p}}$ is a ring, so we see that the image of a^{-1} in $A(\mathfrak{p}) \otimes K_{\mathfrak{p}}$ must be equal to $a_N^{-1} T^{-N} \sum_{i \geq 0} f^i$.

Write $f = f_1 + f_2 \in \hat{R}_{\mathfrak{p}}[r/T, T]$ where f_1 consists of those terms of f which have both positive T -degree and coefficients which are \mathfrak{p} -units. Since $v_{\mathfrak{p}}(a_{i_0}) = 0$, we see that $f_1 \neq 0$. Since $a^{-1} \in A \otimes K$, there exists $\lambda \in R$ such that $\lambda a^{-1} \in A$. Write

$$\lambda a^{-1} = \sum_{j \in \mathbb{Z}} \alpha_j T^j \in F^+ = A.$$

So in $A(\mathfrak{p}) \otimes K_{\mathfrak{p}}$ we find the equality

$$\lambda \left(a_N^{-1} T^{-N} \sum_{i \geq 0} f^i \right) = \lambda \left(a_N^{-1} T^{-N} \sum_{i \geq 0} (f_1 + f_2)^i \right) = \sum_j \alpha_j T^j,$$

and hence

$$\lambda \sum_{i \geq 0} f^i = \lambda \sum_{i \geq 0} (f_1 + f_2)^i = a_N \sum_{j \in \mathbb{Z}} \alpha_j T^{j+N}.$$

Since a_N is a \mathfrak{p} -unit, the series $\sum_{i \geq 0} f^i \in W(\mathfrak{p})$ has coefficients in $\hat{R}_{\mathfrak{p}}$, and

$$v_{\mathfrak{p}}(\alpha_j) = v_{\mathfrak{p}}(a_N \alpha_j) \geq v_{\mathfrak{p}}(\lambda).$$

Thus we may assume that $\lambda \in R$ is a \mathfrak{p} -unit. Write

$$\sum_{i \geq 0} f_1^i =: \sum_{j \geq 0} \beta_j T^j \in 1 + TR[1/a_N][[T]].$$

As all the coefficients of f_2 have \mathfrak{p} -adic valuation strictly positive, we obtain

$$v_{\mathfrak{p}}(\alpha_{j-N}) > 0 \text{ for } j < 0 \quad \text{and} \quad v_{\mathfrak{p}}(a_N \alpha_{j-N} - \lambda \beta_j) > 0 \text{ for } j \geq 0.$$

Let R' be the normalisation of the quotient R/\mathfrak{p} , and set

$$K' = \text{Frac}(R'), \quad A' = \frac{R'[[x, y]]}{(xy - \bar{r})}.$$

For any $b \in R[1/a_N]$, write \bar{b} for its image in K' ; note that this makes sense since $a_N \notin \mathfrak{p}$. In $K'[[T]]$ we find

$$\bar{\lambda} \sum_{i \geq 0} \bar{f}_1^i = \bar{a}_N \sum_{j \geq 0} \bar{\alpha}_{j-N} T^j \in R'[[T]].$$

We are done if we can show that an element of $K'[[T]]$ of the form $\bar{\lambda} \sum_{i \geq 0} \bar{f}_1^i$ can never be contained in $R'[[T]] \subseteq K'[[T]]$. By localising and completing at some generic point of the codimension-1 subscheme $V(\mathfrak{q} R') \rightarrow \text{Spec}(R')$, we complete the proof by applying Lemma 4.4 to the polynomial $\bar{a}_N + \bar{a}_{N+1}T + \cdots + \bar{a}_m T^{m-N}$. \square

Lemma 4.4. *Let R be a complete discrete valuation ring with fraction field K , valuation v and uniformiser π . Let $f(T) = \sum_{i=0}^n a_i T^i \in R[T]$ be a polynomial with $a_0 \neq 0$. Assume that the Newton polygon of f has a side of slope < 0 (in particular that f has positive degree). Write $f = a_0(1 - \tilde{f})$ with $\tilde{f} \in TK[T]$, and*

$$\sum_{i \geq 0} \tilde{f}^i := \sum_{j \geq 0} b_j T^j \in K[[T]].$$

Then $\liminf_j v(b_j) = -\infty$.

Proof. Up to replacing K by a finite extension, we may assume that f splits in K , so all the slopes of the Newton polygon of f are integers. Write $g(T) = f(\pi^\rho T)$ where $-\rho < 0$ denotes the smallest slope of the Newton polygon of f . Then the smallest slope of the Newton polygon of g is 0. We need to show that, if we write $\tilde{g} = \tilde{f}(\pi^\rho T) \in R[T]$, then there are infinitely many coefficients in $\sum_{i \geq 0} \tilde{g}^i$ which are invertible. Observe that $g = a_0(1 - \tilde{g})$ and that g has a side of slope zero with one end over T^0 . Write k for the residue field of R , and h for the image of \tilde{g} in $k[[T]]$. It is enough to show that the series $\sum_{i \geq 0} h^i \in k[[T]]$ is not a polynomial in T . Equivalently, we need to show that the inverse of $1 - h$ in $k[[T]]$ is not a polynomial. This is clear since $\deg(h) > 0$ (since the Newton polygon of g has a side of slope zero starting from T^0). \square

5. Alignment is equivalent to flatness of closure of unit section in the relative Picard space

In this section we return to our usual situation of a generically-smooth semistable curve C over a base S . We want to understand more about the closure of the unit section in the relative Picard scheme of C/S . In fact, we will show that the closure of the unit section is flat (even étale) over S if and only if C/S is aligned. In Section 6 we will relate the flatness of the closure of the unit section to the existence of Néron models.

5.1. Test curves. In this preliminary section, we will define ‘test curves’ in S which we will later use to detect flatness.

Definition 5.1. Given a scheme S , a point $s \in S$ and an open subscheme $U \subseteq S$, a *non-degenerate trait in S through s* is a morphism $\varphi: X \rightarrow S$ where X is the spectrum of a discrete valuation ring, and such that φ maps the closed point of X to s and the generic point of X to a point in U .

Lemma 5.2 (Non-degenerate traits exist). *Let S be a noetherian scheme, $s \in S$ a point, and $U \subseteq S$ a dense open subscheme. Then there exists a non-degenerate trait X in S through s .*

Proof. This is a special case of [10, Proposition 7.1.9]. □

5.2. A natural map to the closure of the unit section. The point of this section is to use the data of an integer weighting of the vertices of the reduction graph Γ_s to define a map from S to the closure of the unit section in the relative Picard scheme $\text{Pic}_{C/S}^{[0]}$, in the case where C/S is aligned.

Definition 5.3. Let (S, s) be a strictly henselian local scheme, and let $C \rightarrow S$ be semi-stable and smooth over some schematically dense open subscheme $U \subseteq S$ (so that all labels of the graph Γ_s are non-zero-divisors in $\mathcal{O}_S(S)$ by Lemma 2.10). An *(integer) vertex labelling of the graph $\Gamma_s = (V, E, \text{ends})$* is a function $m: V \rightarrow \mathbb{Z}$.

Let $\varphi: T \rightarrow S$ be a non-degenerate trait through s . We say a vertex labelling m is *T -Cartier* (better: φ -Cartier) if for every edge e in Γ_s with endpoints v_1 and v_2 , we have that $m(v_1) - m(v_2)$ is divisible by the thickness (see [23, Definition 10.3.23]) of the singular point in the special fibre of C_T corresponding to e .

Given a fibral Weil divisor D on C_T , we define a vertex labelling m of Γ_s by attaching to a vertex v the multiplicity of D along the component corresponding to v .

Lemma 5.4. *In the above notation, let m be a vertex labelling. Then m is T -Cartier if and only if there exists a fibral Cartier divisor D on C_T whose associated labelling is m .*

Proof. This is [28, p. 15]. □

If we suppose that C/S is aligned at s , then from a T -Cartier vertex labelling m we will construct a map from S to the closure of the unit section in the Picard space $\text{Pic}_{C/S}^{[0]}$. To give

such a map is to give a line bundle on C/S , trivial over U . To construct this, we will first construct a Weil divisor on C/S , then prove that it is Cartier and hence defines a line bundle.

When the base is a trait and C is regular, one often constructs Cartier divisors on C by taking irreducible components of the special fibre. If the base is a trait and C is not regular but is semistable, then a similar procedure can be performed taking care of multiplicities; see, e.g., [8]. In our situation, the base S is regular (but maybe of dimension greater than 1) and C is semistable (but perhaps not regular). We will now carry out the analogous construction of Cartier divisors.

Definition 5.5. We retain the above notation, and write $S = \operatorname{Spec} A$. Let $a \in A$ be an element which is not a zerodivisor and not a unit. Let $E(a)$ denote the set of edges of Γ_s whose labels b have the property that the ideal aA is a power of the ideal bA . Let $\Gamma_s(a)$ denote the graph obtained from Γ_s by deleting all the edges in $E(a)$ (note that Γ_s and $\Gamma_s(a)$ have the same vertex sets).

Suppose also that S is regular. We will now give a recipe to associate a Cartier divisor on C to any connected component of $\Gamma_s(a)$. We will then use these Cartier divisors to construct line bundles which will play an essential role in the proof of our main results. First, we need the notion of a ‘specialisation map’ on labelled graphs. The construction of this map is carried out in detail (and also much greater generality) in [18, §5], so we only give an outline here.

Recall that S is strictly henselian local. Let $p \in S$ be any point. Then all singular points on the fibre C_p are rational points, and moreover all irreducible components of C_p are geometrically irreducible. To prove this, we note that $\operatorname{Sing}(C/S) \rightarrow S$ is finite and unramified, and hence is a disjoint union of closed immersions by [31, Tag 04GL], so the assertion on the singular points holds. For the geometric irreducibility, we note that there exists a section through the smooth locus of every irreducible component.

Because of this, we naturally obtain a dual graph Γ_p of C , even though the residue field of p may not be separably closed. Similarly, we find that the labels on the graph Γ_p can be defined in an analogous way to that used in the usual case, and moreover that these labels can be taken to be principal ideals of the Zariski local ring of S at p (whereas usually they live in the étale local ring).

Given two points p, q of S with $p \in \overline{\{q\}}$, we have a canonical injective map

$$\operatorname{sp}: \mathcal{O}_{S,p} \rightarrow \mathcal{O}_{S,q}.$$

If we replace each label on the graph Γ_p by its image under the map sp , then we obtain a new graph with labels in $\mathcal{O}_{S,q}$. Some of the labels may be units; contract all such edges. Then the resulting graph is naturally isomorphic to the labelled graph Γ_q .

Write η_1, \dots, η_n for the generic points of $\operatorname{Spec} A/a$; these are height 1 prime ideals in S . Write m_i for the order of vanishing of a at η_i . Let H be a connected component of the graph $\Gamma_s(a)$. For each $1 \leq i \leq n$, let $Z_i^1, \dots, Z_i^{r_i}$ denote the vertices of Γ_{η_i} which are images under the specialisation map of vertices in H . We can view each Z_i^j as a prime Weil divisor on C .

Definition 5.6. Define a Weil divisor $\operatorname{div}(a; H)$ by

$$\operatorname{div}(a; H) = \sum_{i=1}^n m_i \sum_{j=1}^{r_i} Z_i^j.$$

We call such divisors *primitive fibral Cartier divisors*; the name will be justified by the following lemma.

Lemma 5.7. *In the above setup, we have that $\operatorname{div}(a; H)$ is a Cartier divisor on C .*

Proof. We may assume S is complete. Since C is noetherian, integral and normal, it is enough by Lemma 5.8 to check that D is Cartier at closed points of C . Since $C \rightarrow S$ is proper it maps closed points to closed points, and so it is enough to check that D is Cartier at all closed points of the closed fibre of $C \rightarrow S$.

The result is clear at smooth points in the fibre (since smooth over regular implies regular), so it is enough to look at points of C corresponding to edges in Γ_S . Let e be an edge of Γ_S . If e has exactly 0 or 2 endpoints in H (for example, if e is not in $E(a)$), then this is easy; in the first case the divisor $\operatorname{div}(a; H)$ is cut out near the point corresponding to e by a unit, and in the second case by the function a . To see this second case, we first reduce (by looking at the local ring on S at one of the η_i) to the case where S is a DVR, η_i is the closed point, and a is a power of a uniformiser π , so $a = \pi^{m_i}$. Let e be an edge with both endpoints in H , and let Z and Z' be the irreducible components of C_{η_i} corresponding to those endpoints, so that near the point e the divisor $\operatorname{div}(a; H)$ is given by $m_i(Z + Z')$. Now $\operatorname{ord}_Z(\pi) = 1$, for example because the maximal ideal of the local ring at Z is generated by π , and the same for Z' . Then near the point e we have that $\operatorname{div}(a; H) = m_i(Z + Z') = m_i \operatorname{div} \pi = \operatorname{div} a$.

It remains to treat the case where the edge e is in $E(a)$, and has one vertex v in H and one vertex v' not in H .

Let b denote the label of e , so the completed local ring at the closed point e on C is isomorphic to

$$R := A[[x, y]]/(xy - b),$$

and $a = b^n$ for some $n > 0$. Setting $X = \operatorname{Spec} R$, let $\varphi: X \rightarrow C$ be the map given by choosing such an isomorphism. Without loss of generality, assume that the function x vanishes on φ^*v and that y vanishes on φ^*v' (here we are thinking of v and v' both as vertices of Γ_S and as irreducible components of C_S , to avoid excessive notation).

If we can show that $\varphi^* \operatorname{div}(a; H) = \operatorname{div}_X x^n$ then we are done by Proposition 2.9. Note that both $\varphi^* \operatorname{div}(a; H)$ and $\operatorname{div}_X x^n$ are Weil divisors on X supported over $\operatorname{div}_S a$. Consider one of the irreducible components of $\operatorname{div}_S a$, say η_1 . Then the specialisation map $\operatorname{sp}: \Gamma_S \rightarrow \Gamma_{\eta_1}$ does not map v and v' to the same vertex, since (by construction) no edges in $E(a)$ are contracted by the map sp . Write $V = \operatorname{sp} v$ and $V' = \operatorname{sp} v'$, so V and V' are irreducible components of the fibre C_{η_1} .

Writing \bar{V} and \bar{V}' for the Zariski closure of V and V' , respectively, in C , we find that $\bar{V} \supseteq v$ and $\bar{V}' \supseteq v'$ (and also $\bar{V} \not\supseteq v'$ and $\bar{V}' \not\supseteq v$), and the same holds after pulling back along φ . From this we see that x^n does not vanish on $\varphi^* \bar{V}'$, and y^n does not vanish on $\varphi^* \bar{V}$. Now on X we have

$$\operatorname{div} a = \operatorname{div} b^n = \operatorname{div} x^n + \operatorname{div} y^n,$$

so we see that x^n vanishes on $\varphi^* \bar{V}$ with the same multiplicity as a (and also y^n vanishes on $\varphi^* \bar{V}'$ with the same multiplicity as a). This shows that

$$\varphi^* \operatorname{div}(a; H) = \operatorname{div}_X x^n$$

as required. □

Lemma 5.8. *Let X be a normal integral locally noetherian scheme, and let D be a Weil divisor on X . Suppose that for every closed point x of X , the pullback of D to $X_x = \operatorname{Spec} \mathcal{O}_{X,x}$ is Cartier. Then D is Cartier on X .*

Proof. Since X is locally noetherian, every point of X specialises to a closed point of X . As such, it suffices to give for every closed point $x \in X$ an open neighbourhood U_x of x such that the pullback of D to U_x is Cartier.

Pick a closed point $x \in X$. By assumption, there is an element $f_x \in \operatorname{Frac} \mathcal{O}_{X,x}$ such that on X_x , we have that

$$D|_{X_x} = \operatorname{div}_{X_x} f_x.$$

Now let V be any affine open neighbourhood of x , and compare the divisors $D|_V$ and $\operatorname{div}_V f_x$. Clearly the point x is not contained in the support of the difference between these divisors. Let U_x be the open neighbourhood of x obtained by deleting the support of that difference. Then $D|_{U_x}$ is principal, cut out by f_x . \square

Our key existence result for Cartier divisors is Lemma 5.13. Before proving it we need some results on graph theory.

5.3. Graph theory 1: Cartier functions. Let $G = (V, E, \text{ends})$ be a finite connected graph, with edges labelled by positive integers (in practice, these will be thicknesses of singularities). Analogously to Definition 5.3, we say a function $f: V \rightarrow \mathbb{Z}$ is *Cartier* if given any edge $e \in E$ (with endpoints v_1 and v_2), we have that the label of e divides $f(v_1) - f(v_2)$. We say a function $f: V \rightarrow \mathbb{Z}$ is *elementary-Cartier* if there exists an integer $n \geq 1$ such that

- (i) the function f is constant on connected components of the graph obtained from G by deleting the set of edges $\{e \in E \mid \text{label}(e) \text{ divides } n\}$;
- (ii) f takes values in $n\mathbb{Z}$.

Note that every elementary-Cartier function is Cartier.

Lemma 5.9. *Every Cartier function on V can be written as an integer linear combination of elementary-Cartier functions.*

Proof. Write $V = \{v_1, \dots, v_n\}$, and let f be a Cartier function. We will write f as an integer linear combination of elementary-Cartier functions. Note that, by taking $n = 1$, every constant function is elementary-Cartier. We may thus assume that $f(v_1) = 0$.

Suppose f vanishes on v_1, \dots, v_r for some $1 \leq r < n$. We will construct a sum g of elementary-Cartier functions such that $f - g$ vanishes on v_1, \dots, v_{r+1} , which will yield the claim by induction.

We are assuming that f vanishes on v_1, \dots, v_r , and so the same must hold for g . If we write G' for the graph obtained from G by contracting v_1, \dots, v_r to a single vertex, then clearly f is a lift (along the ‘contraction’ map $G \rightarrow G'$) of a function on G' , and one also sees that any (elementary-)Cartier function on G' lifts to an (elementary-)Cartier function on G . As such, we are reduced to the case where $r = 1$.

We have that $f(v_1) = 0$, and we need to construct an integer linear combination g of elementary-Cartier functions such that $g(v_1) = 0$ and $g(v_2) = f(v_2)$. We need a bit more

notation, First, let Ω denote the set of prime numbers p such that $p \mid \text{label}(e)$ for some edge e of G , i.e.,

$$\Omega = \{p \text{ prime} \mid \text{there exists } e \in E \text{ such that } p \text{ divides } \text{label}(e)\}.$$

Given a path γ in G , we write $\gcd(\gamma)$ for the greatest common divisor of the labels of edges in γ . A *cut* is a set C of edges of G such that v_1 and v_2 lie in different connected components of the graph obtained from G by deleting the edges in C (we write it $G \setminus C$). If C is a cut, we write $\text{lcm}(C)$ for the lowest common multiple of labels of edges in C .

Since f is Cartier, we see that for any path γ in G from v_1 to v_2 , we have that

$$\gcd(\gamma) \mid f(v_2),$$

hence

$$\text{lcm}_\gamma \gcd(\gamma) \mid f(v_2),$$

where the ‘lcm’ is over paths γ from v_1 to v_2 .

Let $p \in \Omega$ be a prime, and let G_p be a labelled graph which has the same underlying graph as G , but with labels given by $\text{ord}_p \text{label}_G$. Let

$$m_p := \text{ord}_p \text{lcm}_\gamma \gcd(\gamma) = \max_\gamma \text{ord}_p \gcd(\gamma) = \max_\gamma \min_{e \in \gamma} \ell_p(e),$$

where γ is as above. Applying ‘max-flow min-cut’ to the graph G_p (see Remark 5.10) yields

$$m_p = \min_C \max_{e \in C} \ell_p(e),$$

where the min is now over cuts. We re-write this as

$$m_p = \min_C \text{ord}_p \text{lcm}(C).$$

Let C_p be a cut achieving this minimum. We have constructed C_p as a cut of G_p , but we can also think of it as a cut of G ; from now on we will do so. Set $L_p = \text{lcm}(C_p)$ where we take the labels in G . Define a function $g_p: V \rightarrow \mathbb{Z}$ to take the value L_p on vertices in the same connected component of $G \setminus C_p$ as v_2 and 0 elsewhere. The function g_p is elementary-Cartier as a function on G since the G -labels of edges in C_p divide L_p .

Note that

$$\gcd\{L_p \mid p \in \Omega\} \mid f(v_2)$$

since

$$\text{ord}_p(\text{lcm}(C_p)) = m_p = \text{ord}_p(\text{lcm}_\gamma \gcd(\gamma)) \leq \text{ord}_p f(v_2).$$

Hence there exist integers n_p such that

$$f(v_2) = \sum_{p \in \Omega} n_p L_p.$$

Then setting

$$g = \sum_{p \in \Omega} n_p g_p,$$

we see that g takes the required values. \square

Remark 5.10. The version of ‘max-flow min-cut’ applied in the above proof is a trivial variation of a standard result. It is easy to prove but needs some care to extract from the literature, so we state and prove it here.

Let H be a finite graph with edges labelled by non-negative integers, and h_1 and h_2 distinct vertices. The *flow* along a path γ from h_1 to h_2 is the minimum of the labels along the edges in the path, and the *value* of a cut C separating h_1 and h_2 is the maximum of the labels on edges in the cut. Then *max-flow min-cut* states that the maximum over paths γ of the flow along γ (‘max flow’) equals the minimum over cuts C of the value of C (‘min cut’).

The relation $\min \text{ cut} \geq \max \text{ flow}$ is obvious. For the converse, let V' denote the set of vertices in H which can be reached from h_1 by traversing only edges with label at least the min cut. If $h_2 \in V'$ then we are done, and if not then the set of edges with one end in V' and the other end not in V' is a cut, contradicting the definition of V' .

Lemma 5.11. *Let (S, s) be a regular strictly henselian local scheme, C/S a generically-smooth semistable curve, and $\varphi: T \rightarrow S$ a non-degenerate trait through s . Let $a \in \mathcal{O}_{S,s}$. Define $\Gamma_s(a)$ as in Definition 5.5. Let $\mathcal{G}_1, \dots, \mathcal{G}_n$ denote the connected components of $\Gamma_s(a)$. Let $0 = c_1, c_2, \dots, c_n$ be integers, and let $m: V \rightarrow \mathbb{Z}$ be the vertex labelling given by assigning to a vertex v of Γ_s the integer c_i such that v is contained in \mathcal{G}_i . Assume that the c_i are chosen such that m is T -Cartier.*

*Then there exists a Cartier divisor D on C , trivial over the generic point of S , and such that the vertex labelling of φ^*D is m .*

Proof. We begin by deleting all self-loops from G (as they have no impact on the functions we want to construct).

Write $S = \text{Spec } A$ and write ord_T for the normalised valuation on T . Given an edge $e \in E(a)$, we know by definition that some power of the label of e is equal to the ideal aA . Since S is factorial, there exists $\alpha \in A$ such that for every edge $e \in E(a)$, the label of e is a power of αA .

We are interested in functions on the vertices of Γ_s which are constant on the \mathcal{G}_i . These are canonically the same as functions on the vertices of the graph G obtained from Γ_s by contracting each \mathcal{G}_i to a point – in what follows we will sometimes not distinguish between these, but it will hopefully be clear from the context which is intended.

Before proceeding with the proof, we will define various labellings on the edges and vertices of the graph G .

Labellings on edges of G :

- ℓ_A the usual edge labelling taking values in the monoid of principal ideals of A (but restricted to edges in G).
- ℓ_T taking values in \mathbb{Z} , defined by sending an edge e to $\text{ord}_T \varphi^* \ell_A(e)$. This coincides with (the restriction to G of) the usual labelling taking values in principal ideals of T , followed by the ord_T map to \mathbb{Z} .
- ℓ_α defined by $\ell_\alpha(e) = \text{ord}_T \ell_A(e) / \text{ord}_T \alpha$. This also takes values in \mathbb{Z} by definition of α .

It is clear that $\ell_T(e) = \ell_\alpha(e) \text{ord}_T \alpha$ for every e .

Labellings on vertices of G :

- The labelling (corresponding to) m , taking values in \mathbb{Z} .
- Now G is connected, and for every edge e from u to v we have that

$$\ell_T(e) \mid m(u) - m(v)$$

and also $\text{ord}_T \alpha \mid \ell_T(e)$. Using also that m takes the value 0 on some vertex, we see that for every vertex v , the integer $m(v)$ is divisible by $\text{ord}_T \alpha$. We then define a new vertex labelling m' by $m'(v) = m(v)/\text{ord}_T \alpha$, taking values in \mathbb{Z} . We see immediately that m' is Cartier with respect to the labelling ℓ_α .

Using Lemma 5.9, we decompose m' into a sum of vertex labellings which are elementary-Cartier with respect to ℓ_α . We can then assume without loss of generality that there exists a positive integer r and a set of vertices H of G such that m' takes the value 0 outside H and the value r on H , and such that for all edges e in G with exactly one end in H we have that $\ell_\alpha(e) \mid r$.

Define $D := \text{div}(\alpha^r; H)$. This is Cartier by Lemma 5.7. We then claim that the divisor D on C_T is the divisor that we seek, i.e. that the vertex labelling of $\varphi^* D$ is $m = m' \text{ord}_T \alpha$.

This is easy: first, it is clear that $\varphi^* \text{div}(\alpha^r; H)$ corresponds to the 0 vertex labelling outside H . Let $h \in H$ be any vertex, and write η_h for the generic point of the irreducible component of the special fibre of C_T corresponding to h . Then by the proof of Lemma 5.7 we see that $\varphi^* \text{div}(\alpha^r; H)$ is cut out by α^r near η_h . Finally, we verify that

$$\text{ord}_T \alpha^r = r \text{ord}_T \alpha = m'(h) \text{ord}_T \alpha = m(h)$$

as required. □

5.4. Graph theory 2: Achievable functions. Let $G = (V, E, \text{ends})$ be a connected graph, with edges labelled by a set L (i.e., a map $l: E \rightarrow L$ is given). We say a function $f: V \rightarrow \mathbb{Z}$ is *pre-achievable* if there exists a subset B of E such that

- every edge in B has the same label;
- the function f is locally constant on connected components of the graph obtained by deleting every edge in B from G .

We say a function $f: V \rightarrow \mathbb{Z}$ is *achievable* if it can be written as a (finite) sum of pre-achievable functions. One can show without too much difficulty that, if labels l are constant on circuits in G , then every function $f: V \rightarrow \mathbb{Z}$ is achievable. However, this is not enough for us; we also need to take into account an analogue of the condition that a vertex labelling be Cartier. In order to translate this notion into graph theory (and forget temporarily the geometric origins of the problem), we will add an additional labelling to the graph G ; namely, we will label each edge $e \in E$ by a positive integer $n(e)$ (in addition to the labelling by symbols in L given earlier). As before, we say that a function $f: V \rightarrow \mathbb{Z}$ is *Cartier* if for every edge $e \in E$ with endpoints v_1 and v_2 , we have that $n(e)$ divides the difference $f(v_1) - f(v_2)$. We say a function f is *Cartier-achievable* if it can be written as a sum of Cartier pre-achievable functions.

Lemma 5.12. *In the setup of the previous paragraph, suppose that the labels in L are constant on circuits in G (we do not require that the integer labellings are constant). Then every Cartier function $f: V \rightarrow \mathbb{Z}$ is Cartier-achievable, i.e. can be written $f = \sum_i f_i$ with each f_i Cartier and pre-achievable.*

Suppose moreover that we pick a vertex v_0 such that $f(v_0) = 0$. Then it is possible to choose the f_i such that $f_i(v_0) = 0$ for all i .

Before proving Lemma 5.12, we briefly discuss the notion of a 2-vertex-connected component of a graph G . Suppose that G has no loops. Observe that any edge of a graph G is contained in a 2-vertex-connected subgraph (namely itself), and further that for any two 2-vertex-connected subgraphs H_1 and H_2 containing the same edge e , we have that $H_1 \cup H_2$ is also 2-vertex-connected. As such, any edge of G is contained in a unique maximal 2-vertex-connected subgraph. In particular, if G contains no isolated vertices, then G is the union of its maximal 2-vertex-connected subgraphs.

Proof of Lemma 5.12. We immediately reduce to the case where G has no loops. We will inductively pick a sequence of subgraphs

$$v_0 \in H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$$

and Cartier pre-achievable functions f_1, \dots, f_n on G such that $f|_{H_i} = \sum_{j=1}^i f_j|_{H_i}$ for all i . It is clear by induction on i that $f_i(v_0) = 0$ for all i .

Pick $H_1 = G_1$ to be the 2-vertex-connected component of G containing v_0 . Then pick H_{i+1} to be the union of H_i with some 2-vertex-connected component G_{i+1} of G such that $G_{i+1} \cap H_i$ consists of exactly one vertex. Since G is connected, we see that this procedure eventually exhausts the whole of G .

Define f_i by

$$f_i(v) = \begin{cases} f(v) - \sum_{j=1}^{i-1} f_j(v) & \text{if } v \in H_i, \\ f(v') - \sum_{j=1}^{i-1} f_j(v') & \text{if } v \notin H_i, \end{cases}$$

where v' is chosen to be the unique vertex in H_i such that v and v' are connected in the graph obtained from G by removing every edge in H_i .

We will now verify that each function f_i is Cartier and pre-achievable, and so that f is Cartier-achievable. We will first treat f_1 , then the general case by induction.

Since f is Cartier, we see that f_1 satisfies the Cartier divisibility condition for every edge in H_1 . For every edge e not contained in H_1 , the function f_1 takes the same value at both endpoints of e , and hence the divisibility condition is trivially satisfied, so f_1 is Cartier. Since the labels on G are constant on circuits, and H_1 is 2-vertex-connected, it follows that the labels are constant on $H_1 = G_1$. Define B_1 to be the set of edges in G_1 , then it is clear that f_1 is constant on connected components of $G \setminus B_1$.

Now we treat the general case. Choose $2 \leq i \leq n$, and suppose that f_j is Cartier for all $j < i$. Now the function

$$f'_i(v) = \begin{cases} f(v) & \text{if } v \in H_i, \\ f(v') & \text{if } v \notin H_i \end{cases}$$

(with v' defined as above) is Cartier by the same argument as in the f_1 case, and hence f_i is Cartier since a sum of Cartier functions is Cartier.

It remains to check that f_i is pre-achievable. Recalling that G_i is a maximal 2-vertex-connected component and satisfies $H_i = G_i \cup H_{i-1}$, we define B_i to be the set of edges in G_i . As before, the labels are constant on B_i because the labels are constant on circuits and G_i is 2-vertex-connected. Consider the connected components of $G \setminus B_i$. One of these components is H_{i-1} . Let C be any other component. Then we see by construction of f_i that it takes a constant value on C ; the value it takes is exactly $f_i(v_C)$ where v_C is the unique vertex contained in both C and G_i . It remains to show that f_i takes a constant value on H_{i-1} . Indeed, it takes the value 0; pick any vertex $v \in H_{i-1}$, then

$$f_i(v) = f(v) - f_{i-1}(v) - \sum_{j=1}^{i-2} f_j(v) = f(v) - \left(f(v) - \sum_{j=1}^{i-2} f_j(v) \right) - \sum_{j=1}^{i-2} f_j(v) = 0. \quad \square$$

5.5. Proof of the main existence result. The key existence result for Cartier divisors is the following lemma.

Lemma 5.13. *Let (S, s) be a regular strictly henselian local scheme, and C/S a semi-stable curve smooth over a dense open $U \subseteq S$. Assume that C/S is aligned at s . Let $\varphi: T \rightarrow S$ be a non-degenerate trait through s . Let m denote a T -Cartier vertex labelling of Γ_s which takes the value 0 at at least one vertex. Then there exists a Cartier divisor D on C/S , trivial over the generic point of S , such that m is the vertex labelling corresponding to $\varphi^* D$.*

Proof. We will introduce two new labellings on Γ_s . First we will also label the edges by integers; for this, we will make a map from $\mathcal{O}_S/\mathcal{O}_S^\times$ to $\mathbb{N}_0 \cup \{\infty\}$, by sending an element a to $\text{ord}_T \varphi^* a$, and then compose our usual labelling by elements of $\mathcal{O}_S/\mathcal{O}_S^\times$ with this map. Note that all of our edges will be labelled by positive integers.

Next, let L denote the quotient of $\mathcal{O}_S/\mathcal{O}_S^\times$ by the equivalent relation saying that $[a] = [b]$ if and only if $a^n = ub^m$ for some unit $u \in \mathcal{O}_S^\times$ and integers $m, n > 0$. Composing our ‘usual’ labelling of Γ_s by $\mathcal{O}_S/\mathcal{O}_S^\times$ with the natural map $\mathcal{O}_S/\mathcal{O}_S^\times \rightarrow L$ gives a labelling of the edges of Γ_s by L .

We now have three distinct (but related) labellings on the edges of Γ_s . We will denote the original labelling by elements of $\mathcal{O}_S/\mathcal{O}_S^\times$ by ℓ_{orig} , the labelling by elements of the quotient L by ℓ_L , and the labelling by integers by ℓ_T (since it is the only one which depends on T).

We then see that a vertex labelling of Γ_s is T -Cartier if and only if it is Cartier with respect to ℓ_T in the graph-theory sense (cf. Lemma 5.4).

By the assumption that C/S is aligned, the graph Γ_s has the property that the ℓ_L labels are constant on circuits. As such, by Lemma 5.12 the T -Cartier vertex labelling m can be written as a sum of pre-achievable T -Cartier functions (with respect to the two labellings ℓ_L and ℓ_T) all of which take the value 0 at some vertex.

It is then enough to show the following: let $m: V \rightarrow \mathbb{N}_0$ be a function which is pre-achievable Cartier with respect to the two labellings ℓ_L and ℓ_T in the sense of the definition above Lemma 5.12, and which takes the value 0 somewhere. Then there exists a Cartier divisor D on C , trivial over the generic point of S , and such that m is the vertex labelling corresponding to $\varphi^* D$.

To show this, we will apply Lemma 5.11. In order to do so, we first need to pick an element $a \in \mathcal{O}_S$. Well, since m is pre-achievable there exists by definition an element $l \in L$ such that f is constant on the connected components of the graph obtained from Γ_s by deleting

every edge with label equal to l . Then there exists $a \in \mathcal{O}_S$ such that for every edge e of Γ_s with $\ell_L(e) = l$, we have that a is a power of $\ell_{\text{orig}}(e)$. We fix such an a , and we will apply Lemma 5.11 using that a .

Now in order to apply Lemma 5.11 we need to check two things:

- (i) The function m is T -Cartier.
- (ii) The function m is constant on connected components of the graph $\Gamma_s(a)$.

For (i), as remarked above, we see from the construction that a vertex labelling of Γ_s is T -Cartier if and only if it is Cartier in the graph-theory sense (this depends only on ℓ_T). For (ii), the function m is by definition constant on connected components of the graph obtained from Γ_s by deleting every edge e such that $\ell_L(e) = l$. By definition of a we have that a is a power of $\ell_{\text{orig}}(e)$ whenever $\ell_L(e) = l$. \square

5.6. Non-aligned curves, and non-existence of certain Cartier divisors.

Lemma 5.14. *Let S be a regular scheme, $U \subseteq S$ a dense open subscheme, and let $\pi: C \rightarrow S$ be a semistable curve, smooth over U . Assume that C/S is not aligned at some $s \in S$. Then there exist a non-degenerate trait $\varphi: T \rightarrow S$ through s and a Cartier divisor D on C_T , trivial over the generic point, such that there does not exist a Cartier divisor E on C/S , trivial over U and such that φ^*E is linearly equivalent to D .*

In fact, the proof will show that for *every* non-degenerate trait T through s a Cartier divisor on C_T as in the statement can be found.

Example 5.15. Before giving the proof, it may be helpful to consider an example. Suppose $S = \text{Spec } k[[u, v]]$ with k separably closed, and take C/S so that the dual graph of the central fibre is a 2-gon with labels u and uv (so C/S is not aligned and C is not regular). Let \mathfrak{C}_0 and \mathfrak{C}_1 be the irreducible components of the central fibre. Choose T to be any non-degenerate trait through the closed point s of S , set $d = \text{ord}_T uv$ and let D be the Cartier divisor $d\mathfrak{C}_1$ on C_T . Then there is an obvious candidate for E as a Weil divisor: namely, the fibre of C over $(u = 0)$ breaks up into two irreducible components, and we could set $E = dX_1$ where X_1 is the component containing \mathfrak{C}_1 . The problem is that E is not Cartier at the non-regular point (where the local equation is $xy = uv$); X_1 corresponds to the prime ideal (x, u) assuming x vanishes on X_1 , and $(x, u)^d$ is not principal in

$$k[[u, v]][[x, y]]/(xy - uv).$$

To go from this to a proof takes more work – for example, perhaps we could have made a different choice of Weil divisor E , and in any case we must consider more general curves. The key ingredient is Theorem 4.1.

Proof of Lemma 5.14. We may assume that S is complete, local and has separably closed residue field. Write s for the closed point, and $\mathfrak{C}_0, \dots, \mathfrak{C}_n$ for the irreducible components of C_s . Then for each i , C/S has a section σ_i passing through the smooth locus of \mathfrak{C}_i .

Suppose we are given two distinct components which meet at some non-smooth point p – for simplicity of notation assume they are \mathfrak{C}_1 and \mathfrak{C}_2 – whose completed local ring in C is

isomorphic to

$$R := \mathcal{O}_{S,s}[[x, y]]/(xy - a)$$

for some $a \in \mathcal{O}_{S,s}$. A Cartier divisor E on C is given locally near p by an element $r \in (\text{Frac } R)^\times$. Let $f_1, f_2 \in \text{Frac}(\mathcal{O}_{S,s})^\times$ be such that $\text{div } f_i = \sigma_i^* E$. Note that the divisor $\sigma_i^* E$ is independent of the choice of section σ_i through the smooth locus of \mathbb{C}_i – indeed, the Cartier divisor $f^* \sigma_i^* E$ is equal to E along the smooth locus of \mathbb{C}_i . If we assume that E is trivial over U , we claim there exists $\delta \in \mathbb{Z}$ such that $f_1 = a^\delta f_2$ (up to multiplication by units in $\mathcal{O}_{S,s}$).

Our next aim is to establish the claim. To fix notation, let us assume that x vanishes on \mathbb{C}_1 , and y vanishes on \mathbb{C}_2 . Firstly, we reduce to the case where E is effective by adding to E the pullback of some effective Cartier divisor on S . By Theorem 4.1 (which applies by Lemma 2.10) we find $b \in \mathcal{O}_{S,s}$, $u \in R^\times$ and $m, n \in \mathbb{Z}_{\geq 0}$ such that $r = bux^m y^n$.

Since the function y is invertible at the generic point η_1 of \mathbb{C}_1 , it follows that at η_1 the Cartier divisor E is also defined by bx^m , and hence by ba^m (recalling $xy = a$ and y is invertible at η_1). Hence we have that $\sigma_1^* E = \text{div } ba^m$. A similar argument shows that $\sigma_2^* E = \text{div } ba^n$, so $\sigma_1^* E = \sigma_2^* E + \text{div } a^{m-n}$, establishing the claim.

By the assumption that C/S is non-aligned at s , there is a circuit in the labelled graph Γ_s with the property that the labels around the circuit do not satisfy any non-trivial multiplicative relation. Let $\mathbb{C}_0, \dots, \mathbb{C}_N$ be the vertices around such a circuit (in order), and write a_i for the label of the edge joining \mathbb{C}_i to \mathbb{C}_{i+1} (working modulo $N + 1$).

Choose any non-degenerate trait $\varphi: T \rightarrow S$ through s . Since φ is non-degenerate, we see that $\varphi^* a_0$ and $\varphi^* a_1$ are both non-zero. Since a_0 and a_1 both vanish at the closed point s (since S is local), we see that $\varphi^* a_0$ and $\varphi^* a_1$ are both non-units. Hence we have that $\text{ord}_T \varphi^* a_0 > 0$ and $\text{ord}_T \varphi^* a_1 > 0$.

Without loss of generality, let us assume that the labels a_0 and a_1 are not related – more precisely, that no non-trivial multiplicative relation holds between the principal ideals a_0 and a_1 of $\mathcal{O}_{S,s}$. We will now define a suitable divisor D on C_T : let d denote the product of all thicknesses of all singular points of $C_T = C \times_{S,\varphi} T$, and set $D = d\mathbb{C}_1$. We see by Lemma 5.4 that D is Cartier. It is clear that D has degree zero over U , hence by [5, Proposition 9.1.2] it has degree zero on every fibre. Now suppose a divisor E as in the hypotheses does exist. We see that $(\sigma_0)_T^* D = 0$ and we may assume that $\sigma_0^* E = 0$. With this assumption, note that if $\varphi^* E$ is linearly equivalent to D then it is equal to D (since $\pi_* \mathcal{O}_C = \mathcal{O}_S$ by [9, Exercise 9.3.11], and so the only principal divisors supported on the special fibre are multiples of the fibre itself). Let $f_0, \dots, f_N \in \mathcal{O}_{S,s}$ be such that $\text{div } f_i = \sigma_i^* E$. If two elements \star and \star' of $\mathcal{O}_{S,s}$ differ by multiplication by a unit, we write $\star \sim \star'$, so in particular we have $f_0 \sim 1$. From the above claim we know that $f_1 \sim a_0^{d_0} f_0$ for some $d_0 \in \mathbb{Z}$, so we have

$$\text{ord}_T(\varphi^* a_0)^{d_0} = \text{ord}_T \varphi^* f_1 = \text{ord}_T \varphi^* \sigma_1^* D = d,$$

and so $d_0 = d / \text{ord}_T a_0$ and hence $f_1 \sim a_0^{d / \text{ord}_T a_0} f_0$. Similar calculations show that

$$f_2 \sim a_1^{-d / \text{ord}_T a_1} f_1 \quad \text{and} \quad f_2 \sim f_3 \sim \dots \sim f_N \sim f_0.$$

Combining these, we find that $a_0^{d / \text{ord}_T a_0} \sim a_1^{d / \text{ord}_T a_1}$, which is a non-trivial multiplicative relation between a_0 and a_1 (since $(\text{ord}_T \varphi^* a_0)(\text{ord}_T \varphi^* a_1)$ divides d), contradicting our assumptions. \square

5.7. Alignment is equivalent to flatness of the closure of the unit section in the relative Picard space. We recall the definition and some basic properties of the relative Picard space as in [5].

Definition 5.16 ([5, Definition 8.1.2]). Let S be a scheme, and X/S an S -scheme. We define the *relative Picard functor of X over S* to be the fppf-sheaf associated to the functor

$$\mathrm{Sch}_S^{\mathrm{op}} \rightarrow \mathrm{Sets}, \quad T \mapsto \mathrm{Pic}(X \times_S T).$$

We denote it by $\mathrm{Pic}_{X/S}$. Tensor product of line bundles gives a group structure on $\mathrm{Pic}(X \times_S T)$ for each T , yielding a factorisation of this functor via abelian groups.

Given a test scheme T/S , it is often useful to have a concrete way to represent elements of $\mathrm{Pic}_{X/S}$. The simplest way to do this is to assume that $\pi: X \rightarrow S$ is proper and of finite presentation, that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ universally, and that π has a section, in which case the natural sequence

$$0 \rightarrow \mathrm{Pic}(T) \rightarrow \mathrm{Pic}(X \times_S T) \rightarrow \mathrm{Pic}_{X/S}(T) \rightarrow 0$$

is exact [5, Proposition 8.1.4]. One can use rigifcators to make this even more explicit [5, §8.1]. Note that the condition that $\pi_*\mathcal{O}_X = \mathcal{O}_S$ holds universally is true in our situation by [9, Exercise 9.3.11].

There are numerous results on representability of the relative Picard functor by a scheme or an algebraic space. We are interested in the relative Picard functor of C/S where C/S is a semistable curve. In this case $\mathrm{Pic}_{C/S}$ is a smooth quasi-separated algebraic space over S by [5, Theorems 8.3.1 and 9.4.1], which apply since $C \rightarrow S$ has reduced and connected geometric fibres, and so is cohomologically flat in dimension 0. Moreover, the fibre-wise connected component of the identity (denoted $\mathrm{Pic}_{C/S}^0$) is a separated S -scheme [5, Theorem 9.4.1]. Note that over a geometric point $s \in S$ the fibre $(\mathrm{Pic}_{C/S}^0)_s$ coincides with the subfunctor of $\mathrm{Pic}_{C_s/s}$ consisting of line bundles having degree zero on every irreducible component of the fibre C_s .

Suppose in addition to the above that C/S is smooth after pullback along some schematically dense quasi-compact open immersion $U \hookrightarrow S$. Another important subfunctor of $\mathrm{Pic}_{C/S}$ is the space $\mathrm{Pic}_{C/S}^{[0]}$ of ‘line bundles of total degree zero’. We define $\mathrm{Pic}_{C/S}^{[0]}$ to be the scheme-theoretic closure in $\mathrm{Pic}_{C/S}$ of $\mathrm{Pic}_{C_U/U}^0$. Its name is justified by the fact that it parametrises line bundles of degree zero on every fibre. To see this, note that the subfunctor (temporarily denoted P^0) of $\mathrm{Pic}_{C/S}$ consisting of line bundles having degree zero on every fibre is an open and closed subfunctor, and clearly contains $\mathrm{Pic}_{C_U/U}^0$ and hence $\mathrm{Pic}_{C/S}^{[0]}$. Since it is open in $\mathrm{Pic}_{C/S}$, we have that P^0 is smooth over S , and so by [13, Théorème 11.10.5 (ii)] we have that $\mathrm{Pic}_{C_U/U}^0$ is schematically dense in P^0 , and we are done.

Theorem 5.17. *Let S be a regular scheme, $U \subseteq S$ a dense open subscheme, and C/S a semistable curve, smooth over U . Then the following are equivalent:*

- (i) C/S is aligned.
- (ii) The closure of the unit section in $\mathrm{Pic}_{C/S}^{[0]}$ is étale over S .
- (iii) The closure of the unit section in $\mathrm{Pic}_{C/S}^{[0]}$ is flat over S .

One of the referees pointed out that it should be possible to replace the assumption that S is regular in the above theorem by ‘for all smooth morphisms $T \rightarrow S$ of relative dimension 1, the scheme T is locally factorial’, at the cost of lengthening some proofs.

Proof. Since all these properties are local on the target, we immediately reduce to the case where S is strictly henselian local. Then the smooth locus of C/S admits sections through every connected component of every fibre, so C/S is projective and every irreducible component of every fibre is geometrically irreducible, so that $\text{Pic}_{C/S}$ is a scheme by [5, Theorem 8.2.2].

First we prove (i) \Rightarrow (ii). Suppose that C/S is aligned. Write $\text{clo}(e)$ for the closure of the unit section in $\text{Pic}_{C/S}^{[0]}$, and let $p \in \text{clo}(e)$ be a point. By Lemma 5.18, it is enough to construct a section from S through p in $\text{clo}(e)$. First, let $\varphi: T \rightarrow \text{clo}(e)$ be a non-degenerate trait through p ; so the generic point of T maps to $\text{clo}(e)_U$, and the closed point maps to p (such a T exists by Lemma 5.2). Composing with the structure map from $\text{clo}(e)$ to S , we obtain a non-degenerate trait $\varphi_S: T \rightarrow S$ and a semistable curve C_T/T by pullback. Then the map φ gives a line bundle \mathcal{L} on C_T/T which is trivial over φ_S^*U . To show that $\text{clo}(e)$ admits a section over S through p , we will construct a line bundle $\overline{\mathcal{L}}$ on C/S such that $\varphi_S^*\overline{\mathcal{L}} = \mathcal{L}$ and such that $\overline{\mathcal{L}}$ is trivial over U .

Choosing a rational section of \mathcal{L} , trivial over φ_S^*U , we in turn obtain a Cartier divisor D on the relative curve C_T , with the property that the restriction of D to the generic fibre is zero. We will construct a Cartier divisor \overline{D} on C/S which pulls back to D over T , then define $\overline{\mathcal{L}} = \mathcal{O}_C(\overline{D})$.

From D we obtain a T -Cartier vertex labelling on Γ_S as in Definition 5.3. Without loss of generality, we may assume that this Cartier vertex labelling takes the value 0 somewhere. Then by Lemma 5.13 (here we use that S is regular) we find a Cartier divisor \overline{D} on C as required. This shows (i) \Rightarrow (ii).

The implication (ii) \Rightarrow (iii) is clear.

Finally, we prove (iii) \Rightarrow (i), or rather $\neg(\text{iii}) \Rightarrow \neg(\text{i})$. Suppose then that C/S is not aligned; we will show that $\text{clo}(e)$ is not flat. By Lemma 5.14 (which again uses that S is regular), we find a non-degenerate trait T in S through s and a Cartier divisor D on C_T , zero over the generic point of T , and such that there does not exist a Cartier divisor \overline{D} on C/S which pulls back to a divisor linearly equivalent to D . Now $\mathcal{O}_{C_T}(D)$ gives a map $T \rightarrow \text{Pic}_{C/S}^{[0]}$, and it maps the generic point of T to a point in the unit section. Since $\text{clo}(e)$ is closed in $\text{Pic}_{C/S}^{[0]}$ and the generic point of T lands in $\text{clo}(e)$, we see that the image of T is contained in $\text{clo}(e)$. Write t for the closed point of T and $\varphi: T \rightarrow \text{clo}(e)$ for the given map.

Suppose now that $\text{clo}(e)$ is flat over S ; we will derive a contradiction. By Lemma 5.18 we find a section σ of $\text{clo}(e) \rightarrow S$ through $\varphi(t)$. This section corresponds to a Cartier divisor \overline{D} on C/S . This divisor is zero over U . Since $\text{clo}(e)_U \rightarrow U$ is an isomorphism, we find that the generic point of T also maps to the image of σ . Since T is reduced, we deduce that $\varphi: T \rightarrow \text{clo}(e)$ factors via σ . As such, we find that the pullback of \overline{D} to T is linearly equivalent to the divisor D , a contradiction. \square

Lemma 5.18. *Let S be a noetherian scheme and $U \subseteq S$ dense open. Let $f: X \rightarrow S$ be a morphism of schemes locally of finite type and which is an isomorphism over U , and such that $f^{-1}U$ is schematically dense in X . Let $x \in X$ be a point. The following are equivalent:*

- (i) *f is étale at x .*
- (ii) *f is flat at x .*
- (iii) *There exists an open neighbourhood V of $f(x)$ in S and a section $\sigma: V \rightarrow X$ through x .*

Proof. (i) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii): Shrinking, we may assume f is surjective and of finite type, and both X and S are affine, so f is also separated. Then by [24, Lemma 2.0] we find that f is an isomorphism, in particular, it has a section through x .

(iii) \Rightarrow (i): Shrinking, we may again assume X and $S = V$ are affine, so f is separated. The image of S in X is a closed subscheme (by separatedness of X/S), and contains the schematically dense subscheme $f^{-1}U$. Hence the image of S in X is the whole of X , so f is an isomorphism and hence étale. \square

6. Flatness of closure of the unit section in the Picard space is equivalent to existence of a Néron model

Lemma 6.1 (Néron models satisfy smooth descent). *Let S be a scheme, $U \hookrightarrow S$ a schematically dense quasi-compact open immersion, and $f: S' \rightarrow S$ a smooth surjective morphism. Let A/U be an abelian scheme. Suppose f^*A has a Néron model N' over S' . Then A has a Néron model N over S , and moreover $f^*N = N'$.*

The author would like to thank Kęstutis Česnavičius for pointing out that some separateness hypotheses in an earlier version of this lemma were unnecessary.

Proof. A Néron model is unique if it exists, by the universal property. The pullback of U is schematically dense in S' by [13, Théorème 11.10.5 (ii)]. Moreover, the pullback of a Néron model along a smooth morphism is again a Néron model. Combining this with the effectivity of descent for algebraic spaces [31, Tag 0ADV], we deduce that N' descends to a (smooth, separated) algebraic space N/S . We need to check that the descended object N satisfies the Néron mapping property. Write $S'' = S' \times_S S'$ and $q: S'' \rightarrow S$. Let $T \rightarrow S$ be a smooth morphism, and $g: T_U \rightarrow A$ a U -morphism. By descent, it is enough to check that there is a unique S'' morphism $q^*T \rightarrow q^*N$ extending $q^*g: q^*T_U \rightarrow q^*N$. Now $q^*T \rightarrow S''$ is smooth, in particular flat, so by [13, Théorème 11.10.5 (ii)] again we have that $q^*T_U \rightarrow q^*T$ has scheme-theoretic closure equal to q^*T . Moreover, $q^*N \rightarrow S''$ is separated, so the conclusion holds by [31, Tag 084N]. \square

Theorem 6.2. *Let S be a locally noetherian scheme, $U \subseteq S$ a schematically dense open subscheme, and $f: C \rightarrow S$ a semistable curve which is smooth over U . Write e for the unit section in $\text{Pic}_{C/S}^{[0]}$, and $\text{clo}(e)$ for the closure of e in $\text{Pic}_{C/S}^{[0]}$ (or equivalently in $\text{Pic}_{C/S}$). Write J for the jacobian of the smooth proper curve C_U/U .*

(i) *If C is regular and $\text{clo}(e)$ is flat over S , then a Néron model for J exists.*

(ii) *If a Néron model for J exists, then $\text{clo}(e)$ is étale over S .*

Remark 6.3. (i) It is not really necessary for C to be regular; it would for example suffice to have C_T locally factorial for every smooth morphism $T \rightarrow S$ (which is probably a weaker condition, even if $C \rightarrow S$ is smooth).

(ii) A slightly different approach to proving Theorem 6.2 (i) (suggested to the author by Kęstutis Česnavičius) would be to apply the theory of parafactorial pairs [14, §21.13].

(iii) By the universal property of the Néron model there is a canonical map from $\mathrm{Pic}_{C/S}^0$ to the fibrewise-connected-component-of-identity of the Néron model, which we denote by \mathcal{N}^0 . By [16, IX, Proposition 3.1 (e)] (which applies to algebraic spaces, see discussion below) this map is an open immersion, and hence by [15, VI_A 0.5] it is an isomorphism. Now $\mathrm{Pic}_{C/S}^0$ is a scheme (see [6, Lemme 4.3] or [5, Theorem 9.4.1]), and hence so is \mathcal{N}^0 .

Moreover, the identity component of $\mathrm{Pic}_{C/S}$ (and hence of the Néron model) is a quasi-projective scheme, by [26, Remarque XI 1.3 (c), Théorème XI 1.13].

(iv) The étaleness of the closure of the unit section extends a result of Raynaud [27] from the case where S is a trait, but under much more restrictive hypotheses on C/S .

Proof. Note that $\mathrm{clo}(e)$ is a subgroup space of $\mathrm{Pic}_{C/S}^{[0]}$. For (i), we first observe that the quotient of $\mathrm{Pic}_{C/S}^{[0]}$ by $\mathrm{clo}(e)$ exists as an algebraic space since $\mathrm{clo}(e)$ is flat over S ; we denote this quotient by \mathcal{N} . It is in a canonical way a group algebraic space over S . We will show that \mathcal{N} is the Néron model of J . Combining Lemma 6.1 with the fact that C/S admits sections étale locally (since the smooth locus of C/S meets every fibre), we may assume that C/S has a section.

Note that $J = \mathrm{Pic}_{C/S}^{[0]} \times_S U = \mathcal{N}_U$. Since \mathcal{N} is separated (as its unit section is a closed immersion), the uniqueness part of the Néron mapping property is automatic; we need to show existence. Let $T \rightarrow S$ be a smooth morphism of spaces, and $T_U \rightarrow \mathcal{N}_U$ any S -morphism. Let $T' \rightarrow T$ be an étale morphism, with T' a scheme. By base change, we obtain a morphism $T'_U \rightarrow \mathrm{Pic}_{C/S}^{[0]} \times_S U$.

Since $T' \rightarrow S$ is smooth and C is regular, we see that $T' \times_S C$ is regular [31, Tag 036D]. Since C/S admits a section, the Picard functor coincides with the rigidified Picard functor (rigidified along that section), so in particular there exists a line bundle \mathcal{F} on $T'_U \times_U C_U$ such that the map $T'_U \rightarrow \mathrm{Pic}_{C/S}^{[0]} \times_S U$ is given by that line bundle. Let D be a Cartier divisor on $T'_U \times_U C_U$ such that $\mathcal{O}(D) \cong \mathcal{F}$.

Write $D = \sum_i n_i p_i$ where the p_i are prime Weil divisors. Let \bar{p}_i denote the scheme-theoretic closure of p_i in $T' \times_S C$, and write $\bar{D} = \sum_i n_i \bar{p}_i$. A-priori this \bar{D} is a Weil divisor, but by regularity it is in fact Cartier, so that $\mathcal{O}(\bar{D})|_U = \mathcal{O}(D) \cong \mathcal{F}$. Let $\tilde{\mathcal{F}} := \mathcal{O}(\bar{D})$. Then $\tilde{\mathcal{F}}|_U = \mathcal{F}$. This sheaf $\tilde{\mathcal{F}}$ defines a morphism $T' \rightarrow \mathrm{Pic}_{C/S}^{[0]}$, and by composition a morphism $T' \rightarrow \mathcal{N}$, which coincides with the original map $T'_U \rightarrow \mathcal{N}_U$ upon restriction to U . Finally, we must descend this to a map $T \rightarrow \mathcal{N}$, but this follows from the uniqueness part of the Néron mapping property. This proves the existence part of the Néron mapping property.

To complete the proof of (i), we must show that \mathcal{N} is smooth over S . We know that $\mathrm{Pic}_{C/S}^{[0]}$ is smooth and is an fppf cover⁴⁾ of \mathcal{N} , so we deduce that \mathcal{N} is smooth.

For (ii), write \mathcal{N} for the (smooth) Néron model of J . After base change to an étale cover we may assume that $\mathrm{Pic}_{C/S}$ is a scheme (since being étale is even fpqc local on the target by [31, Tag 02YJ]). We have the identity $\mathrm{Pic}_{C/S}^{[0]} \times_S U = \mathcal{N}_U$, and hence by smoothness of $\mathrm{Pic}_{C/S}^{[0]}$ over S and the Néron mapping property, we get a canonical map $\varphi: \mathrm{Pic}_{C/S}^{[0]} \rightarrow \mathcal{N}$.

Suppose for a moment that the map φ is flat. Write K for the kernel of φ ; then the canonical map $K \rightarrow S$ is also flat, and the canonical map $K \rightarrow \mathrm{Pic}_{C/S}^{[0]}$ is a closed immersion since \mathcal{N} is separated (and so K contains $\mathrm{clo}(e)$). Moreover, the preimage of U is schematically

⁴⁾ We define ‘fppf cover’ as in [31, Tag 021L], in particular it need not be quasi-compact. The result we use can also be found in [14, Proposition 17.7.7].

dense in K by [13, Théorème 11.10.5 (ii)], so in fact $K = \text{clo}(e)$, in particular $\text{clo}(e)$ is flat over S . Then by Lemma 5.18 it is étale over S as required.

As such, it suffices to show that φ is flat. By the *critère de platitude par fibres* [31, Tag 05X0] it is enough to show that for every $s \in S$, the fibre φ_s of φ at s is flat. Further, it is enough to show that the restriction $\varphi_s^0: \text{Pic}_{C_s/s}^0 \rightarrow \mathcal{N}_s^0$ to the connected components of identity is flat – one just covers $\text{Pic}_{C_s/s}^{[0]}$ by translates of $\text{Pic}_{C_s/s}^0$, and uses that the former is a scheme by [2].

Write φ^0 for the restriction of φ to the fibrewise-connected components of identity. Clearly φ^0 is an isomorphism over U . As such,⁵⁾ [16, IX, Proposition 3.1 (e)] implies that φ_0 is an open immersion, in particular flat, hence so is φ_s^0 , and we are done. \square

Corollary 6.4. *Let S be an excellent scheme, regular in codimension 1, and let $U \subseteq S$ be dense open and regular. Let C/S be a semistable curve which is smooth over U . Then there exists an open subscheme $U \subseteq V \subseteq S$ such that the complement of V in S has codimension at least 2 in S and such that the jacobian of C_U/U admits a Néron model over V , whose identity component is a quasi-projective V -scheme.*

Proof. Removing some closed codimension-2 subscheme from S , we may assume S is regular. Alignment outside codimension 2 is clear from the definition. Existence of the Néron model follows immediately from Theorem 5.17 and Lemma 6.2 once we show that C/S has a regular model outside some codimension-2 subscheme of S ; the latter is Proposition 6.5. \square

Proposition 6.5. *Let S, U, C be as in the statement of Corollary 6.4. Then there exist*

- *an open subscheme $U \subseteq V \subseteq S$ whose complement has codimension at least 2,*
- *a modification (proper birational map) $\tilde{C} \rightarrow C_V$ which is an isomorphism over U ,*

such that \tilde{C} is regular and $\tilde{C} \rightarrow V$ is semistable.

Proof. Throwing away codimension-2 loci, we reduce immediately to the case where S is regular, and $S \setminus U$ is strict normal-crossings divisor. By [21, Lemma 3.2], there exists a projective modification of semistable curves $\tilde{C} \rightarrow C$ which is an isomorphism over U , and such that \tilde{C} is regular outside some locus of codimension 3 in \tilde{C} . The image of that locus has codimension at least 2 in S ; throwing it away, we are done. \square

7. Non-aligned implies multiples of sections do not lift even after proper surjective base change

We have shown that the jacobian of a non-aligned semistable curve does not admit a Néron model, even after an alteration of the base. On the other hand, Gabber's lemma [6]

⁵⁾ Here we apply the result to a morphism of spaces, when the reference given proves it for schemes. It turns out that the same proof works for spaces.

shows that any abelian scheme admits a semi-abelian prolongation after alteration⁶⁾ of the base.

As a result, it is perhaps reasonable to hope that the situation is better if one does not attempt to construct a Néron model but rather considers only one section at a time. Given a scheme S , a dense open subscheme U , an abelian scheme A/U , and a section $\sigma \in A(U)$, one can ask whether there exists a semi-abelian prolongation \mathcal{A}/S such that some multiple of σ extends to a section in $\mathcal{A}(S)$, at least after replacing S by an alteration. A positive answer to this question was incorrectly claimed by the author in a previous version of this paper on arXiv; the error was pointed out by José Burgos Gil. In this section, we provide an example to show that such a semi-abelian prolongation prolonging σ will not in general exist even after proper surjective base change of S .

Construct a stable 2-pointed curve over \mathbb{C} by gluing two copies of $\mathbb{P}_{\mathbb{C}}^1$ at $(0:1)$ and $(1:0)$, and marking the point $(1:1)$ on each copy. Then define C/S to be the universal deformation as a 2-pointed stable curve. Choose coordinates such that $S = \operatorname{Spec} \mathbb{C}[[x, y]]$, and C is smooth over the open subset $U = D(xy)$. Call the sections p and q .

Now the graph over the closed point of S is a 2-gon, with one edge labelled by (x) and the other by (y) . The graph over the generic point of $(x = 0)$ is a 1-gon with edge labelled by (y) , and similarly the graph over the generic point of $(y = 0)$ is a 1-gon with edge labelled by (x) . All other fibres are smooth. In particular, C/S is aligned except at the closed point.

Let J/U denote the jacobian of C_U/U , and write $\sigma = [p - q] \in J(U)$. A *pointed semi-abelian prolongation* consists of

- a proper surjective morphism $f: S' \rightarrow S$,
- a semi-abelian scheme \mathcal{A}/S' ,
- an isomorphism $\mathcal{A}|_{f^{-1}U} \rightarrow f^*J$,
- an integer $n \geq 1$,
- a section $\tau \in \mathcal{A}(S')$,

such that τ extends $n \cdot f_U^* \sigma$, where f_U denotes the restriction of f to $f^{-1}U$.

Proposition 7.1. *No pointed semi-abelian prolongation exists.*

Proof. Suppose a pointed semi-abelian prolongation is given. Let X be a prime divisor on S' lying over the prime divisor (x) of S such that f^*C is not aligned at some geometric point $p \in X$ (such a prime divisor X exists by the easy direction of Lemma 2.22). The graph of f^*C over p is a 2-gon, with one edge labelled by X . Write Z for the label of the other edge. Let $\varphi_i: T_i \rightarrow S'$ be a sequence of non-degenerate traits in S' through P such that $\operatorname{ord}_{T_i} X = 1$ for all i , and such that $\operatorname{ord}_{T_i} Z$ tends to infinity as i tends to infinity. For each i , the pullback $(\varphi_i \circ f)^*J$ admits a Néron model; write σ_i for the extension of the section σ given by the Néron mapping property. Then a simple calculation shows that the order of σ_i in the component group of the Néron model of $(\varphi_i \circ f)^*J$ tends to infinity with i . This contradicts the existence of a pointed semi-abelian prolongation. \square

⁶⁾ The statement in [6] requires a proper surjective morphism rather than just an alteration. However, the same proof yields the result after an alteration by replacing the reference to [7, Definition 4.11, Theorem 4.12] in Lemme 1.6 by a reference to [22, Théorème 16.6]. Alternatively, one can deduce this version from that in [6] by a slicing argument.

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